Computing in Cantor's Paradise With λ_{ZFC}

or

How to Make an Infinitary λ-Calculus in κ Easy Steps

(and Why)

FLOPS 2012

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PLT @ Brigham Young University

Living in Cantor's Paradise

David Hilbert

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- Simple example: $\pi = 16 \tan^{-1}\left(\frac{1}{5}\right) - 4 \tan^{-1}\left(\frac{1}{239}\right)$

Options for Domain Specific Languages

- Option 1: Write theorems and proofs in a proof assistant, extract programs
  - Problem: Re-proving theorems takes a long time!
  - Hurd 2002: Dissertation half comprised of convincing HOL of theorems from early-1900s measure theory
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- ∃n. Option n is a middle ground?

Example: Beautiful Differentiation

- Elliot 2010: Derives an automatic differentiation implementation from this exact specification:

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- How Elliot does it:

1. Reformulates differentiation in terms of *toD* to hide use of *d*

2. Uses differentiation theorems and **`Functor`** and **`Applicative`** instance definitions to derive *d*-free functions

3. Implements using floating-point to approximate reals

Beautiful Derivations

$$
\begin{array}{lll}
& toD\ (sin\ u) & \\
\equiv & liftA_2\ D\ (sin\ u)\ (d\ (sin\ u)) & \text{definition of } toD \\
\equiv & liftA_2\ D\ (sin\ u)\ (d\ u * cos\ u) & d\ (sin\ u) = d\ u * cos\ u \\
\equiv & \lambda x \rightarrow D\ ((sin\ u)\ x)\ ((d\ u * cos\ u)\ x) & \text{definition of } liftA_2 \\
\equiv & \ldots & \\
\equiv & sin\ (toD\ u) &
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\equiv & \ldots & \\
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$$

- Problem: To elegantly derive the implementation, the derivations have to be done in a language that doesn't exist!

Apologies

"We have no implementation of *d*, so this definition of *toD* will **serve as a specification**, not an implementation."

"This **definition is not executable**, however, since *d* is not."

"Every remaining use of *d* is applied to a function whose derivative is known, so we can replace each use.... Now we have **an executable implementation again**."

"Again, this definition can be refactored, followed by replacing the **non-effective [unimplementable] applications** of *d* with known derivatives."

Example: From Bayesian Notation to Pure Racket

- Toronto and McCarthy 2010: Derive implementation of Bayesian modeling language from an exact specification using

$$\begin{aligned} \mathsf{sum}\ \mathsf{f}\ \mathsf{A} &:= \sum_{\omega \in \mathsf{A}} \mathsf{f}\ \omega \\ \mathsf{preimage}\ \mathsf{A}\ \mathsf{f}\ \mathsf{B} &:= \{\mathsf{x} \in \mathsf{A} \mid \mathsf{f}\ \mathsf{x} \in \mathsf{B}\} \end{aligned}$$

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- How we did it:

1. Assume a very powerful lambda calculus ($\boldsymbol{\lambda}_{\mathrm{ZFC}}$) exists
2. Define exact meaning of Bayesian notation in this language
3. Derive implementable approximation, prove convergence

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"[Sketch of $\boldsymbol{\lambda}_{\mathrm{ZFC}}$ features.] **We intend $\boldsymbol{\lambda}_{\mathrm{ZFC}}$** to be contemporary mathematics with well-defined lambdas, or a practical lambda calculus with infinite sets."

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Infinite Sets and Set Operations

λ_{ZFC}

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λ calculus **Infinite Sets and Set Operations** $\boldsymbol{\lambda}_{ZFC}$

- "Computing in Cantor's Paradise With $\boldsymbol{\lambda}_{ZFC}$" realizes this vision

Lambda-ZFC Requirements

- Must be similar to implementation language
  - Higher-order functions and lambdas
  - Computable sublanguage
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  - Apply well-known theorems with only trivial translation
- Should treat values uniformly (all values first-class)
  - Specifically allow lambdas in sets: $\{\lambda x.\, x, \lambda x.\, x + x, ...\}$
  - For minimalism: want to use $\langle a, b \rangle := \{\{a\}, \{a, b\}\}$

Building Lambda-ZFC- From the ZFC Axioms (1)

Axiom	$\boldsymbol{\lambda}^{-}_{\mathrm{ZFC}}$
Empty set. There is a set $\varnothing$ with no elements.	$\varnothing$ (value symbol)

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Cardinality. Every set A has a unique cardinality $\lvert A\rvert$.	$\mathsf{card}\ e$

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Infinity. The language ω of $n ::= 0 \mid n + 1$ is a set, where $0 = \varnothing$ and $n + 1 = n \cup \{n\}$.	ω (value symbol)

Grammar of Finite Terms

- Final $\boldsymbol{\lambda}^{-}_{\mathrm{ZFC}}$ grammar:

$$e \;::=\; x \mid v \mid e\ e \mid \mathsf{if}\ e\ e\ e \mid$$

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- Problem: What should $\mathsf{P}\ \varnothing$ reduce to? $\{\varnothing\}$ isn't a value...

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- Solution 1: Recursively encode expressions as sets using tags (i.e. SICP-style records)
  - $\langle t_\lambda, e \rangle$ (equivalently $\{\{t_\lambda\}, \{t_\lambda, e\}\}$) is a lambda with body e
  - $\langle t_{\text{set}}, A \rangle$ is a set containing the members of A (e.g. the encoding of $\{\varnothing\}$ is $\langle t_{\text{set}}, \{\langle t_{\text{set}}, \varnothing \rangle\}\rangle$)

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  - $\langle t_{\mathrm{set}}, A\rangle$ is a set containing the members of A (e.g. the encoding of $\{\varnothing\}$ is $\langle t_{\mathrm{set}}, \{\langle t_{\mathrm{set}}, \varnothing\rangle\}\rangle$)
- Problem 2: "All the sets" is not a set
- Solution 2: Find a set that acts enough like "all the sets"

All the Sets (1)

- Curious fact: unfolding $\mathcal{P}$ generates the **hereditarily finite** sets

$$\begin{aligned} \mathcal{V}(0) &= \varnothing \\ \mathcal{V}(n+1) &= \mathcal{P}(\mathcal{V}(n)) \qquad \text{successor ordinal } n+1 \end{aligned}$$

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- Curious fact: ω is also a number: the **first countable ordinal**

$$\begin{aligned} 0 &= \varnothing & \omega &= \{0, 1, 2, ...\} \\ 1 &= \{0\} & \omega+1 &= \{0, 1, 2, ..., \omega\} \\ 2 &= \{0, 1\} & \omega+2 &= \{0, 1, 2, ..., \omega, \omega+1\} \\ 3 &= \{0, 1, 2\} & \omega+\omega &= \{0, 1, 2, ..., \omega, \omega+1, \omega+2, ...\} \end{aligned}$$

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- Can define more **limit ordinals** $\omega \cdot \omega$, ω^{ω}, $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$

All the Sets (2)

- Curious fact: unfolding $\mathcal{P}$ actually generates all the sets

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  - Every set first appears in some $\mathcal{V}(\alpha)$; e.g. $\mathbb{R} \in \mathcal{V}(\omega + 11)$

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- Can we stop unfolding at some ordinal and have a set of sets that is closed under set primitives?
  - Yes: $\mathcal{V}(\omega)$ is closed under set primitives; it's called a **Grothendieck universe**

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- Is there a Grothendieck universe that contains ω? Undecidable.

Inaccessible Cardinal Axiom

Axiom (inaccessible cardinal). There exists an ordinal κ such that $\mathcal{V}(\kappa)$ contains ω and is closed under $\mathcal{P}$, $\bigcup$, and replacement.

- Call sets in $\mathcal{V}(\kappa)$ **hereditarily accessible** sets

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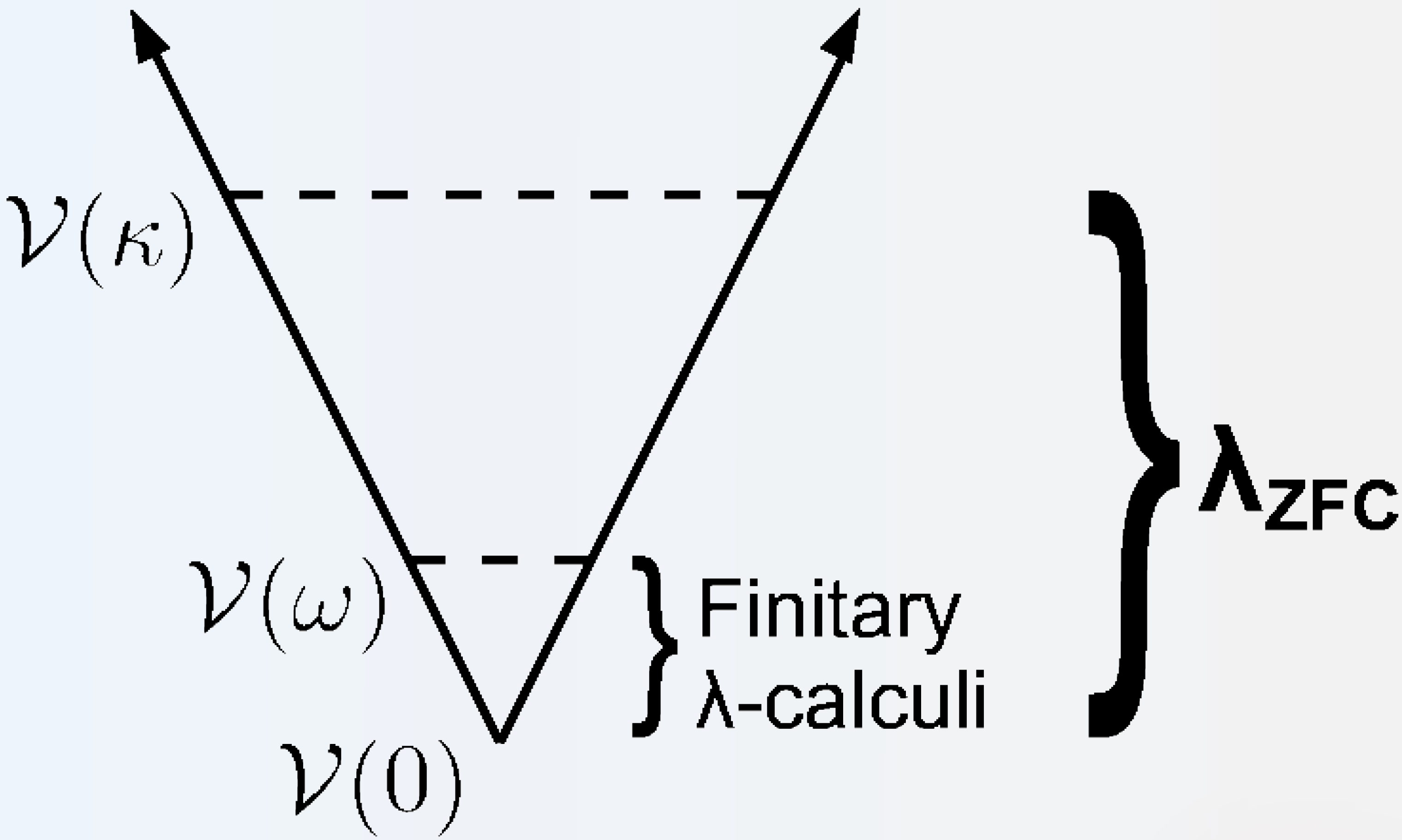
- Uncontroversial extension to ZFC, relatively mild (c.f. HOL, Coq)
- No corresponding $\boldsymbol{\lambda}^{-}_{\mathrm{ZFC}}$ or $\boldsymbol{\lambda}_{\mathrm{ZFC}}$ expression

The Hierarchy of Sets



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- Example: $v ::= \langle t_{\text{set}}, \{v^{*\kappa}\}\rangle$
  - Language is every set in $\mathcal{V}(\kappa)$, recursively tagged

Finite Grammar of Infinite Terms

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$$
\begin{aligned}
e \;&::=\; n \mid v \mid \langle t_{\mathrm{app}}, e, e\rangle \mid \langle t_{\mathrm{if}}, e, e, e\rangle \mid \langle t_{\in}, e, e\rangle \mid \langle t_{\cup}, e\rangle \mid \\
&\quad\;\; \langle t_{\mathrm{take}}, e\rangle \mid \langle t_{\mathcal{P}}, e\rangle \mid \langle t_{\mathrm{image}}, e, e\rangle \mid \langle t_{\mathrm{card}}, e\rangle \mid \langle t_{\mathrm{set}}, \{e^{*\kappa}\}\rangle \\
v \;&::=\; \langle t_{\mathrm{atom}}, t_{\mathrm{false}}\rangle \mid \langle t_{\mathrm{atom}}, t_{\mathrm{true}}\rangle \mid \langle t_{\lambda}, e\rangle \mid \langle t_{\mathrm{set}}, \{v^{*\kappa}\}\rangle \\
n \;&::=\; \langle t_{\mathrm{var}}, 0\rangle \mid \langle t_{\mathrm{var}}, 1\rangle \mid \langle t_{\mathrm{var}}, 2\rangle \mid \cdots
\end{aligned}
$$

Finite Grammar of Infinite Terms

- Final $\boldsymbol{\lambda}_{\text{ZFC}}$ grammar:

$$
\begin{aligned}
e \;\; &::= \;\; n \mid v \mid \langle t_{\text{app}}, e, e \rangle \mid \langle t_{\text{if}}, e, e, e \rangle \mid \langle t_{\in}, e, e \rangle \mid \langle t_{\cup}, e \rangle \mid \\
&\quad\;\; \langle t_{\text{take}}, e \rangle \mid \langle t_{\mathcal{P}}, e \rangle \mid \langle t_{\text{image}}, e, e \rangle \mid \langle t_{\text{card}}, e \rangle \mid \langle t_{\text{set}}, \{e^{*\kappa}\} \rangle \\
\boxed{v \;\; ::=} \;\; & \langle t_{\text{atom}}, t_{\text{false}} \rangle \mid \langle t_{\text{atom}}, t_{\text{true}} \rangle \mid \langle t_{\lambda}, e \rangle \mid \boxed{\langle t_{\text{set}}, \{v^{*\kappa}\} \rangle} \\
n \;\; &::= \;\; \langle t_{\text{var}}, 0 \rangle \mid \langle t_{\text{var}}, 1 \rangle \mid \langle t_{\text{var}}, 2 \rangle \mid \cdots
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- Contains encodings of all the sets in $\mathcal{V}(\kappa)$

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$$

- Contains encodings of all the sets in $\mathcal{V}(\kappa)$
- Computable sublanguage: replace $*\kappa$ with $*\omega$
- Ugly! Write in $\boldsymbol{\lambda}^{-}_{\text{ZFC}}$ with heaps of syntactic sugar; assume transformation to $\boldsymbol{\lambda}_{\text{ZFC}}$ before reduction

Lambda-ZFC's Reduction Semantics

Defines a κ-sized, big-step reduction relation '$\Downarrow$':

$$\frac{}{v \Downarrow v}\ (\text{val}) \qquad \frac{e_f \Downarrow \langle t_\lambda, e_y\rangle \quad e_x \Downarrow v_x \quad e_y[0\backslash v_x] \Downarrow v_y}{\langle t_{\text{app}}, e_f, e_x\rangle \Downarrow v_y}\ (\text{ap}) \qquad \frac{e_c \Downarrow a_{\text{true}} \quad e_t \Downarrow v_t}{\langle t_{\text{if}}, e_c, e_t, e_f\rangle \Downarrow v_t} \quad \frac{e_c \Downarrow a_{\text{false}} \quad e_f \Downarrow v_f}{\langle t_{\text{if}}, e_c, e_t, e_f\rangle \Downarrow v_f}\ (\text{if})$$

$$\frac{e_A \Downarrow v_A \quad V_{\text{set}}(v_A) \quad e_x \Downarrow v_x \quad v_x \in snd(v_A)}{\langle t_\in, e_x, e_A\rangle \Downarrow a_{\text{true}}} \quad \frac{e_A \Downarrow v_A \quad V_{\text{set}}(v_A) \quad e_x \Downarrow v_x \quad v_x \notin snd(v_A)}{\langle t_\in, e_x, e_A\rangle \Downarrow a_{\text{false}}}\ (\text{in})$$

$$\frac{e_A \Downarrow v_A \quad V_{\text{set}}(v_A) \quad \forall v_x \in snd(v_A).\, V_{\text{set}}(v_x)}{\langle t_\cup, e_A\rangle \Downarrow \widehat{\bigcup}(v_A)}\ (\text{union}) \qquad \frac{e_A \Downarrow v_A \quad V_{\text{set}}(v_A)}{\langle t_{\mathcal{P}}, e_A\rangle \Downarrow \widehat{\mathcal{P}}(v_A)}\ (\text{pow})$$

$$\frac{e_A \Downarrow v_A \quad V_{\text{set}}(v_A) \quad e_f \Downarrow \langle t_\lambda, e_y\rangle \quad \widehat{I}(\langle t_\lambda, e_y\rangle, v_A) \Downarrow v_y}{\langle t_{\text{image}}, e_f, e_A\rangle \Downarrow v_y}\ (\text{image}) \qquad \frac{e_A \Downarrow v_A \quad V_{\text{set}}(v_A)}{\langle t_{\text{card}}, e_A\rangle \Downarrow \widehat{C}(v_A)}\ (\text{card})$$

$$\frac{E_{\text{set}}(e_A) \quad \forall e_x \in snd(e_A).\, \exists v_x.\, e_x \Downarrow v_x}{e_A \Downarrow \langle t_{\text{set}}, \{v_x \mid e_x \in snd(e_A) \wedge e_x \Downarrow v_x\}\rangle}\ (\text{set}) \qquad \frac{e_A \Downarrow \langle t_{\text{set}}, \{v_x\}\rangle}{\langle t_{\text{take}}, e_A\rangle \Downarrow v_x}\ (\text{take})$$

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Theorem. **λ**$_{\text{ZFC}}$'s set values and $\langle t_\in, \cdot, \cdot\rangle$ are a model of ZFC-κ.

(i.e. theorems that don't depend on κ are true of **λ**$_{\text{ZFC}}$'s sets)

Expressive Enough?

- Implementable in $\boldsymbol{\lambda}_{\mathrm{ZFC}}$:
  - Bounded logic, pairs, naturals, integers, rationals, arithmetic

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  - Set-theoretic `unfold`, general closure operators, metric universes, limits on metric spaces
  - Measure theory: Borel σ-algebras, arbitrary products of σ-algebras, product measures, Lebesgue measure, Lebesgue integration, conditional probability measures

Example Application: Limit Monad

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Example: $\mathsf{sums} : (\omega \to \mathbb{R}) \to (\omega \to \mathbb{R})$

$$\begin{aligned} \mathsf{sums\ xs} &:= \lambda \mathsf{n.\,if\ (n = 0)\ (xs\ 0)\ ((xs\ n) + (sums\ xs\ (n-1)))} \\ \sum_{\mathsf{n} \in \omega} \mathsf{e} &:\equiv \mathsf{limit\ (sums\ } \lambda \mathsf{n.\,e)} \end{aligned}$$

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Example: $\mathsf{exp\text{-}seq} : \mathbb{R} \to (\omega \to \mathbb{R})$

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Uncomputable Limit Monad

- Defining functions:

$$
\begin{aligned}
\mathsf{return}\ \mathsf{x} &:= \lambda \mathsf{n}.\, \mathsf{x} \\
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- Wish for: limit-free, drop-in replacement for bind
  - Reality says: Sorry Bucko, only under limited conditions
- Step 1: Factor using monad identities and topological theorems

$$\begin{aligned} \mathsf{bind}\ xs\ f &= \mathsf{join}\ (\mathsf{lift}\ f\ xs) \\ \mathsf{lift}\ f\ xs &:= \mathsf{return}\ (f\ (\mathsf{limit}\ xs)) \\ \mathsf{join}\ xss &:= \lambda n.\, \mathsf{limit}\ (\mathsf{flip}\ xss\ n) \end{aligned}$$

Computable Limit Monad

- Step 2: Collapse limits using topological theorems

Identity	**Condition (per-instance)**
$\mathsf{limit}\ (\mathsf{lift}\ \mathsf{f}\ \mathsf{xs}) = \mathsf{limit}\ (\mathsf{f} \circ \mathsf{xs})$	Continuity

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- Limit-free replacement for bind:

$$
\begin{aligned}
\mathsf{lift}'\ \mathsf{f}\ \mathsf{xs} &:= \mathsf{f} \circ \mathsf{xs} \\
\mathsf{join}'\ \mathsf{xss} &:= \lambda \mathsf{n}.\, \mathsf{xss}\ \mathsf{n}\ \mathsf{n} \\
\mathsf{bind}'\ \mathsf{xs}\ \mathsf{f} &:= \mathsf{join}'\ (\mathsf{lift}'\ \mathsf{f}\ \mathsf{xs})
\end{aligned}
$$

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- Prove conditions for use of bind in $\mathsf{exp}_{\mathsf{lim}}$; then redefine

$$\mathsf{exp}_{\mathsf{lim}}\ \mathsf{xs} := \mathsf{bind}'\ \mathsf{xs}\ \mathsf{exp\text{-}seq}$$

Limit Monad Example: Computing π

Machin's formula (1706): $\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right)$

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$$\mathsf{atan\text{-}seq}\ \mathsf{y} := \\ \quad \mathsf{sums}\ \lambda\mathsf{n}.\ \frac{-1^{\mathsf{n}} \times \mathsf{y}^{2\times\mathsf{n}+1}}{2\times\mathsf{n}+1}$$

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Racket

```
(define (atan-seq y)
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atan$_{\text{lim}}$ ys :=
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Proof

Continuity
Unif. conv.

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λ_{ZFC}	Proof	Racket
atan-seq y := sums λn. $\dfrac{-1^{n} \times y^{2\times n+1}}{2 \times n + 1}$		`(define (atan-seq y) (sums (λ (n) (/ (* (expt -1 n) (expt y (+ (* n 2) 1))) (+ (* n 2) 1)))))`
$\mathrm{atan}_{\mathrm{lim}}$ ys := bind ys atan-seq	Continuity Unif. conv.	
$\mathrm{atan}_{\mathrm{lim}}$ ys := bind′ ys atan-seq		`(define (atan-lim ys) (bind ys atan-seq))`

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atan-seq y :=

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λ_{ZFC}

atan_{lim} ys :=
bind ys atan-seq

Proof

Continuity
Unif. conv.

λ_{ZFC}

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Racket

```
(define (atan-lim ys)
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```

λ_{ZFC}

π_{lim} :=

$$(r\ 16) \times_{\text{lim}} (\text{atan}_{\text{lim}}\ (r\ \tfrac{1}{5})) -_{\text{lim}}$$
$$(r\ 4) \times_{\text{lim}} (\text{atan}_{\text{lim}}\ (r\ \tfrac{1}{239}))$$

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λ_{ZFC}	**Proof**	**Racket**

atan-seq y :=
 sums λn. $\dfrac{-1^{n} \times y^{2\times n+1}}{2\times n+1}$
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atan_{lim} ys := bind ys atan-seq	Continuity Unif. conv.	
atan_{lim} ys := bind′ ys atan-seq		`(define (atan-lim ys)` `  (bind ys atan-seq))`
π_{lim} := $(r\ 16) \times_{\text{lim}} (\text{atan}_{\text{lim}}\ (r\ \frac{1}{5})) -_{\text{lim}}$ $(r\ 4) \times_{\text{lim}} (\text{atan}_{\text{lim}}\ (r\ \frac{1}{239}))$	*None*	

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λ_{ZFC}	**Proof**	**Racket**

atan-seq y :=
 sums $\lambda n. \dfrac{-1^n \times y^{2\times n+1}}{2 \times n + 1}$
		`(define (atan-seq y)`
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 $(\text{r}\ 4) \times_{\text{lim}} (\text{atan}_{\text{lim}}\ (\text{r}\ \frac{1}{239}))$
	None	`(define pi-lim`
		`  (-lim (*lim (r 16) (atan-lim (r 1/5)))`
		`        (*lim (r 4) (atan-lim (r 1/239)))))`

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Machin's formula (1706): $\pi = 16 \tan^{-1}\left(\frac{1}{5}\right) - 4 \tan^{-1}\left(\frac{1}{239}\right)$

λ_{ZFC}	Proof	Racket

$\text{atan-seq } y :=$
 $\quad \text{sums } \lambda n. \dfrac{-1^n \times y^{2\times n+1}}{2\times n+1}$
		`(define (atan-seq y)`
		`  (sums (λ (n) (/ (* (expt -1 n)`
		`                    (expt y (+ (* n 2) 1)))`
		`                 (+ (* n 2) 1)))))`

$\text{atan}_{\text{lim}} \text{ ys} :=$ $\quad \text{bind ys atan-seq}$	Continuity Unif. conv.	
$\text{atan}_{\text{lim}} \text{ ys} :=$ $\quad \text{bind}' \text{ ys atan-seq}$		`(define (atan-lim ys)` `  (bind ys atan-seq))`

$\pi_{\text{lim}} :=$
 $\quad (\text{r } 16) \times_{\text{lim}} (\text{atan}_{\text{lim}} (\text{r } \frac{1}{5})) -_{\text{lim}}$
 $\quad (\text{r } 4) \times_{\text{lim}} (\text{atan}_{\text{lim}} (\text{r } \frac{1}{239}))$
	None	`(define pi-lim`
		`  (-lim (*lim (r 16) (atan-lim (r 1/5)))`
		`        (*lim (r 4) (atan-lim (r 1/239)))))`

```
> (real->decimal-string (time (pi-lim 141)) 200)
cpu time: 10 real time: 18 gc time: 0
"3.14159265358979323846264338327950288419716939937510 58[...]"
```

Conclusions and Future Work

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  - Defined the limit monad, defined π in it, derived an implementation, transliterated it into Racket

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- Bonus Questions!
  - How much deep set theory do I need to know to use $\boldsymbol{\lambda}_{\mathrm{ZFC}}$?
  - Can proofs about $\boldsymbol{\lambda}_{\mathrm{ZFC}}$ terms be used in contemporary math?
  - How would one make a call-by-name version of $\boldsymbol{\lambda}_{\mathrm{ZFC}}$?
  - Why is $\mathbb{R} \in \mathcal{V}(\omega + 11)$?