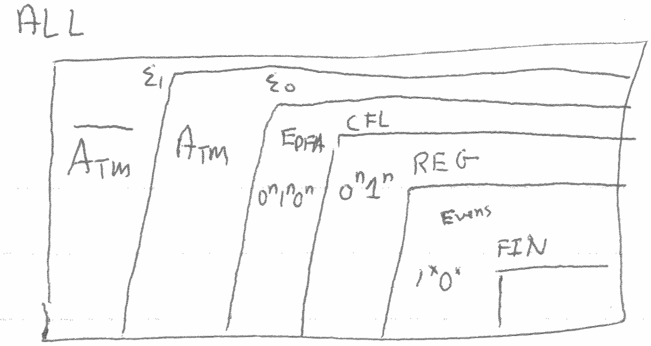
25-1




$\Sigma_1 \neq \Sigma_0$

$A_{TM} \in \Sigma_1 \qquad A_{TM} \notin \Sigma_0$

$\Sigma_1 = ALL?$

$$\underbrace{A \in \Sigma_1}_{P} \wedge \underbrace{\overline{A} \in \Sigma_1}_{} \Rightarrow \underbrace{A \in \Sigma_0}_{Q}$$

Yes(w) = accept if $w \in A$, diverge o.w.    No(w) = accept if $w \notin A$, diverge o.w.

$\hookrightarrow$ Yes'(w) = accept if $w \in A$
reject if $w \notin A$

:= On input w,
while true; { 1. take 1 step on Yes(w)
2. take 1 step on No(w)
if Y says Y, say Y, if N says Y, say N.

$P \Rightarrow Q \quad \Longleftrightarrow \quad \neg Q \Rightarrow \neg P$

$\neg P \vee Q \qquad \neg\neg Q \vee \neg P$

$A \notin \Sigma_0 \Rightarrow \neg(A \in \Sigma_1 \wedge \overline{A} \in \Sigma_1)$

$= A \notin \Sigma_1 \vee \overline{A} \notin \Sigma_1$

$A_{TM} \notin \Sigma_0 \Rightarrow \cancel{A_{TM} \notin \Sigma_1} \vee \underbrace{\overline{A_{TM}} \notin \Sigma_1} \longrightarrow \overline{A_{TM}} \in ALL$ and $\notin \Sigma_1$

$A_{TM} \in \Sigma_1$

$A_{TM}(\langle M, w\rangle) = \cancel{\text{accept if}}$ $w \in L(M)$

~~reject~~ ~~or diverge~~

$\overline{A_{TM}}(x) =$ x is not $\langle M, w\rangle$
= Halting Problem or $x = \langle M, w\rangle$ and $w \notin L(M)$

if M is a decider, M returns reject on w
if M is a recognizer, M diverges on w

25-2/ If $X = Y$, then they have the same size AND the same elements.

When are two sets the same size?

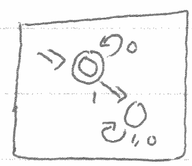
- counting doesn't work
- { cat, dog, human }  M = cat ↦ pika
  { pika, squirtle, bulbasaur }  dog ↦ squir
  human ↦ bulb
- These mappings are bijection: a function from A to B / X to Y
  it must be one-to-one:
  $\forall x \in X.\ \forall x' \in X.\ f(x) = f(x') \to x = x'$
  it must be onto:
  $\forall y \in Y.\ \exists x \in X,\ f(x) = y$

Cantor  Aleph  Hebrew Alpha

"Countable" $\aleph_0$

:= same size as naturals

Naturals  $0^*$



$f: Nat \to 0^*$

$f(n) = 0^n$  $0^n = 0^m \to n = m$ ✓

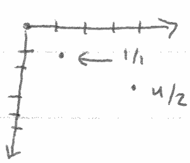
$0^n = 0^y \to n = y$ ✓

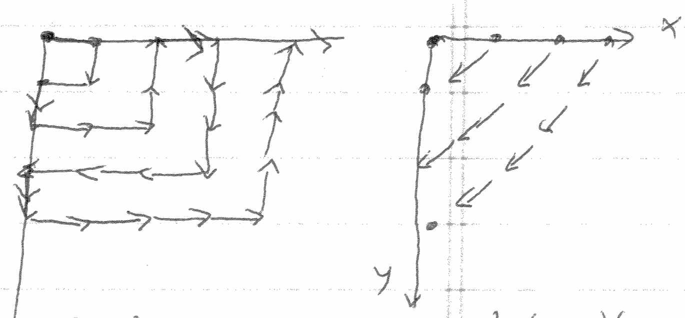
$\aleph$

$X \simeq Y$

Naturals  Evens  $f(n) = 2n$

Naturals $N$  Rationals $= (N \times N)$





Szudik: $f(x,y) = \frac{1}{2}(x+y)(x+y+1)+y$

$N \simeq N \times N$  $N^K \simeq N$

$N \times (N \times N)$ → $N$ → $N$

N, Rationals (Q), Integers (Z)

25-3

Turing Machines are countable

TM $m = (Q, \Sigma, \Gamma, q_0, \delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}, q_a, q_r)$

$Q \to N$, $N$ × ~~…~~ $(2N)^{4N}$, $q_a \to N$, $q_r \to N$

TM $\simeq N^k$ for some $k \simeq N$

$\langle m \rangle = 01 \ldots 01 \ldots 01$ is a number

$\Sigma_1 \simeq N$   $\ulcorner (x, y) \urcorner = \frac{x}{y}$

Rationals $= N \times N$

Real numbers $= 0, -1, \frac{1}{2}, \sqrt{2}, \pi, 0.\bar{3}, e$

Cauchy sequence : real is a converging infinite sequence of rationals

Dedekind cut: it is the infinite set rationals smaller than and bigger than

sequence of digits $= N \to$ digit

Reals in $[0, 1) = N \to$ digit   $0 = \lambda n. 0$

$0.\bar{3} = \lambda n. 3$

Binary in $[0, 1) = N \to \{0, 1\}$   $0.12 = \lambda n.$ if $n=0, 1$; $n = 1, 2$; o.w. $0$

$0 = \lambda n. 0$   $0.5 = \lambda n.$ if $n=0$, $1$, ow $0$

IBS - infinite binary sequence

$\boxed{\text{bin}[0,1)} \simeq N$?

IBS $\simeq N \iff \exists f^{N \to IBS}$. $f$ is onto $\wedge$ $f$ is one-to-one

IBS $\not\simeq N \iff \forall f^{N \to IBS}$. $f$ is not onto or $f$ is not one-to-one

$\forall f : N \to IBS$, $\underline{f \text{ is not onto}}$,   $f$ is onto

$\exists r \in IBS. \forall n \in N. \underline{f(n)} \neq \underline{r}$   $\forall r \in IBS. \exists n \in N. f(n) = r$

given : $f$   $\exists i \in N. f(n)(i) \neq r(i)$   $IBS = N \to \{0, 1\}$

pick : $r$   $f(n) = r$

given : $n$   $\forall i. f(n)(i) = r(i)$

pick : $i = n$

$f(n)(i) \neq r(i)$   $r = IBS = \lambda p. \neg f(p)(p)$

$f(n)(n) \neq r(n)$   $\neq \neg f(n)(n)$

Cantor's Diagonalization Argument