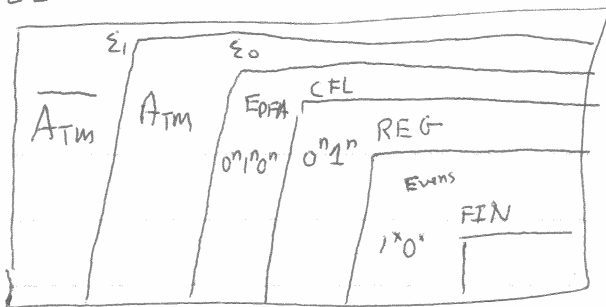


ALL



$$\Sigma_1 \neq \Sigma_0$$

$$ATM \in \Sigma_1, \quad ATM \notin \Sigma_0$$

$$\Sigma_1 = ALL?$$

$$A \in \Sigma_1 \wedge \bar{A} \in \Sigma_1 \Rightarrow A \in \Sigma_0$$

Yes(w) = accept if $w \in A$ diverge o.w. No(w) = accept if $w \notin A$ diverge o.w.

$$\rightarrow Yes'(w) = \begin{cases} \text{accept if } w \in A \\ \text{reject if } w \notin A \end{cases}$$

= On input w , while true:

1. take 1 step on Yes(w)
2. take 1 step on No(w)

 if Y says Y, say Y, if N says N, say N.

$$P \Rightarrow Q \iff \neg Q \Rightarrow \neg P$$

$$\neg P \vee Q \iff \neg \neg Q \vee \neg P$$

$$A \notin \Sigma_0 \Rightarrow \neg (A \in \Sigma_1 \wedge \bar{A} \in \Sigma_1)$$

$$= A \notin \Sigma_1 \vee \bar{A} \notin \Sigma_1$$

$$ATM \notin \Sigma_0 \Rightarrow ATM \notin \Sigma_1 \vee \bar{ATM} \notin \Sigma_1$$

$$ATM \in \Sigma_1 \implies \bar{ATM} \in ALL \text{ and } \notin \Sigma_1$$

$$ATM(\langle M, w \rangle) = \begin{cases} \text{accept if } w \in L(M) \\ \text{reject or diverge otherwise} \end{cases}$$

$$\bar{ATM}(\langle M, w \rangle) = \begin{cases} \text{is not } \langle M, w \rangle \\ \text{Halting Problem on } x = \langle M, w \rangle \text{ and } w \notin L(M) \end{cases}$$

if M is a decider, M returns reject on w

if M is a recognizer, M diverges on w

25-2 / If $X = Y$, then they have the same size AND the same elements.

When are two sets the same size?

- counting doesn't work

$\{ \text{cat, dog, human} \}$ $M = \text{cat} \mapsto \text{pika}$
 $\{ \text{pika, squirtle, bulbasaur} \}$ $\text{dog} \mapsto \text{squirt}$
 $\text{human} \mapsto \text{bulb}$

- These mappings are bijections: a function from A to B it must be one-to-one: X to Y

$\forall x \in X, \forall x' \in X, f(x) = f(x') \rightarrow x = x'$

it must be onto:

$\forall y \in Y, \exists x \in X, f(x) = y$

Cantor
 Aleph
 Hebrew
 Alpha
 \downarrow
 \aleph_0
 "Countable"
 \aleph_0
 := same size as naturals
 \aleph

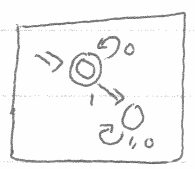
Naturals

$f: \text{Nat} \rightarrow \aleph^*$

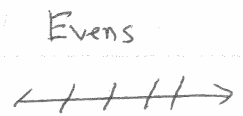
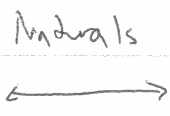
$f(n) = \aleph^n$

$\aleph^n = \aleph^m \rightarrow n = m$ ✓

$\aleph^n = \aleph^y \rightarrow n = y$ ✓

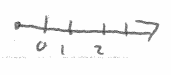


$X \cong Y$

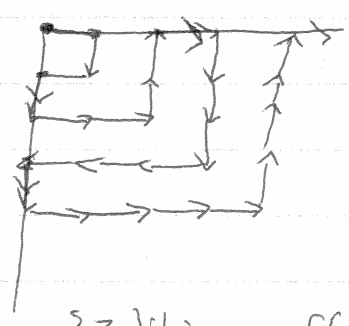
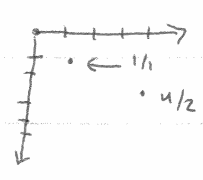


$f(n) = 2n$

Naturals N



Rationals = $(N \times N)$

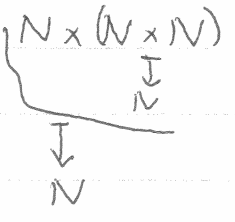


S zuditi:

$f(x, y) = \frac{1}{2}(x+y)(x+y+1) + y$

$N \cong N \times N$

$N^k \cong N$



N , Rationals (\mathbb{Q}), Integers (\mathbb{Z})

25-3/

Turing Machines are countable

$$TM\ m = (Q, \Sigma, \Gamma, q_0, \delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}, q_a, q_r)$$

$$\begin{matrix} \downarrow & & & & & & & & & \downarrow & \downarrow \\ N & \times & N & \times & \cancel{2N} & \times & (2N)^{4N} & & & N & N \end{matrix}$$

TM $\cong N^k$ for some $k \leq N$

$\langle m \rangle = 01 \dots 01 \dots 01$ is a number

$$\Sigma_1 \cong N \quad \overline{(x, y)} = \frac{x}{y}$$

Rationals = $N \times N$

Real numbers = $0, -1, \frac{1}{2}, \sqrt{2}, \pi, 0.\bar{3}, e$

Cauchy sequence: real is a converging infinite sequence of rationals

Dedekind cut: it is the infinite set of rationals smaller than and bigger than

sequence of digits = $N \rightarrow$ digit

Reals in $[0, 1)$ = $N \rightarrow$ digit $0 = \lambda n, 0$

$0.\bar{3} = \lambda n, 3$

Binary in $[0, 1)$ = $N \rightarrow \{0, 1\}$ $0.12 = \lambda n, \text{if } n=0, 1$

$0 = \lambda n, 0$ $0.5 = \lambda n, \text{if } n=0, 1, \text{ or } 0$ $n=1, 2$
 $0, \text{ or } 0$

IBS - infinite binary sequence

$\overline{\text{bin}[0, 1)} \cong N?$

$IBS \cong N \iff \exists f: N \rightarrow IBS$ f is onto & f is one-to-one

$IBS \not\cong N \iff \forall f: N \rightarrow IBS$ f is not onto or f is not one-to-one

$\forall f: N \rightarrow IBS, f$ is not onto.

f is onto

$\exists r \in IBS, \forall n \in N, f(n) \neq r$

$\forall r \in IBS, \exists n \in N, f(n) = r$

given: f $\exists i \in N, f(n)(i) \neq r(i)$ $IBS = N \rightarrow \{0, 1\}$

pick: r $f(n) = r$

given: n $\forall i, f(n)(i) = r(i)$

pick: $j = n$

$f(n)(i) \neq r(i)$ $r \in IBS = \lambda p, \neg f(p)(p)$

$f(n)(n) \neq r(n) \iff \neg f(n)(n)$

Cantor's
Diagonalization
Argument

1. \mathbb{R}^n is a vector space over \mathbb{R} .
2. \mathbb{C}^n is a vector space over \mathbb{C} .

3. \mathbb{R}^n is a vector space over \mathbb{C} if we consider it as a real vector space.
4. \mathbb{C}^n is a vector space over \mathbb{R} if we consider it as a complex vector space.

5. \mathbb{R}^n is a vector space over \mathbb{R} if we consider it as a real vector space.
6. \mathbb{C}^n is a vector space over \mathbb{C} if we consider it as a complex vector space.

7. \mathbb{R}^n is a vector space over \mathbb{R} if we consider it as a real vector space.
8. \mathbb{C}^n is a vector space over \mathbb{C} if we consider it as a complex vector space.

9. \mathbb{R}^n is a vector space over \mathbb{R} if we consider it as a real vector space.
10. \mathbb{C}^n is a vector space over \mathbb{C} if we consider it as a complex vector space.

11. \mathbb{R}^n is a vector space over \mathbb{R} if we consider it as a real vector space.
12. \mathbb{C}^n is a vector space over \mathbb{C} if we consider it as a complex vector space.

13. \mathbb{R}^n is a vector space over \mathbb{R} if we consider it as a real vector space.
14. \mathbb{C}^n is a vector space over \mathbb{C} if we consider it as a complex vector space.

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16. \mathbb{C}^n is a vector space over \mathbb{C} if we consider it as a complex vector space.

17. \mathbb{R}^n is a vector space over \mathbb{R} if we consider it as a real vector space.
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19. \mathbb{R}^n is a vector space over \mathbb{R} if we consider it as a real vector space.
20. \mathbb{C}^n is a vector space over \mathbb{C} if we consider it as a complex vector space.