Function $f : D \Rightarrow R$

Call $f$ with something "like" $D$ but not exactly

$f$ : Animal $\Rightarrow$ number (weight)

$f$ (cats)  $f$(dogs)  Cat $\in$ Ty  Dog $\in$ Ty

$$T = B \mid 1 \mid 0 \mid T \to T \mid T + T \mid T \times T \mid \mu A.T \mid A \mid \forall A.T \mid \exists A.T$$

$$f : \forall A.\ \overbrace{\text{num} \times A}^{\text{representation of Animal}} \Rightarrow \text{num}$$

Animal := num × $\exists A.A$

shared behavior

Animal := $\forall A$. num × $A$ ← specific

Cat := num × str

Dog := num × num

Add "Records" to the language

$$M ::= \dots \mid \langle L = M, \dots \rangle \mid M.L$$

$L$ ::= some set (disjoint from $X$ and $A$)

$$V ::= \dots \mid \langle L = V, \dots \rangle$$

$$E ::= \dots \mid \langle L = V, \dots, L = E, L = M, \dots \rangle$$

$$E[\langle L_0 = V_0, \dots, L_i = V_i, \dots, L_n = V_n \rangle . L_i] \to E[V_i]$$

$$T ::= \dots \mid \langle L : T, \dots \rangle$$

$$\frac{\Gamma \vdash M_0 : T_0 \quad \dots \quad \Gamma \vdash M_n : T_n}{\Gamma \vdash \langle L_0 = M_0, \dots, L_n = M_n \rangle : \langle L_0 : T_0, \dots, L_n : T_n \rangle}$$

$$\frac{\Gamma \vdash M : \langle L_0 : T_0, \dots, L_i : T_i, \dots, L_n : T_n \rangle}{\Gamma \vdash M.L_i : T_i}$$

$\langle x : \text{num}, y : \text{num} \rangle$

$\langle x : \text{num}, y : \text{num}, z : \text{num} \rangle$

17-2/ distance Along X : $\langle x : \text{num} \rangle \to \text{num}$

project to 2d : $\langle x : \text{num}, y : \text{num} \rangle \to \langle x : \text{num}, y : \text{num} \rangle$

$:= \lambda p. \ \langle x = p.x, \ y = p.y \rangle$

Subtyping Relation : $T \leq T$

$\langle L_0 : T_0, \ldots, L_n : T_n \rangle \leq \langle L_0' : T_0', \ldots, L_m' : T_m' \rangle$

if $\{ L_0' : T_0', \ldots, L_m' : T_m' \} \subseteq \{ L_0 : T_0, \ldots, L_n : T_n \}$

$\langle x : \text{num}, y : \text{num}, z : \text{num} \rangle \leq \langle x : \text{num} \rangle$

$$\frac{\Gamma \vdash M : D \to R \qquad \Gamma \vdash N : D' \qquad \overbrace{D' \leq D}^{\text{new and}}}{\Gamma \vdash (M \ N) : R}$$

says "N is compatible"

old relation $\Leftrightarrow$ becomes our new relation $\leq$

$$\frac{\Gamma \vdash D_1 \Leftrightarrow D_2 \qquad \Gamma \vdash R_1 \Leftrightarrow R_2}{\Gamma \vdash (D_1 \to R_1) \Leftrightarrow (D_2 \to R_2)}$$

$\Leftrightarrow$ is an equiv (refl, trans, sym)

$\leq$ is an partial order (refl, trans)

| | $D_1$ on L | $D_1$ on R |
| --- | --- | --- |
| $R_1$ on L | $D_1 \leq D_2$, $R_1 \leq R_2$ ✗ | $D_2 \leq D_1$, $R_1 \leq R_2$ ✓ |
| $R_1$ on R | $D_1 \leq D_2$, $R_2 \leq R_1$ ✗ | $D_2 \leq D_1$, $R_2 \leq R_1$ ✗ |

$$\frac{\Gamma \vdash D_2 \leq D_1 \qquad \Gamma \vdash R_1 \leq R_2}{\Gamma \vdash (D_1 \to R_1) \leq (D_2 \to R_2)}$$

(Animal $\to$ Animal) $\leq$ (Cat $\to$ Cat) ? ✗

let f := ... in

(f garfield).meow()

Animal $\leq$ Cat? ✗

Cat $\leq$ Animal? ✓

(Animal $\to$ Cat) $\leq$ (Cat $\to$ ~~Animal~~) ? ✓

(f garfield).feed()

Liskov Substitution Principle

(Barbara Liskov)

$\langle p : \langle x{:}num, y{:}num, z{:}num \rangle, n{:}str, hp{:}num \rangle$

$\leq \langle p : \langle x{:}num, y{:}num \rangle, n{:}str \rangle$ ?

$\langle L_0 : T_0, \ldots, L_n : T_n \rangle \leq \langle L_0' : T_0', \ldots, L_m' : T_m' \rangle$

if $\{L_0', \ldots, L_m'\} \subseteq \{L_0, \ldots, L_n\}$

and $(L_i = L_j') \Rightarrow (T_j' \leq T_i)$

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Class $C$ with methods $(m_i : D_i \to R_i)$

and fields $(F_j : T_j)$

$\langle m_i : O_C \times D_i \to R_i, \ \ldots \rangle = C$

$O_C = \exists A.\, A$ (internally: $\langle F_j : T_j, \ldots \rangle$)

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$f : \underbrace{\text{Animal}}_{:= \langle M_i : \ldots \rangle} \to num$

class Point { public int x,y } ; $\quad$ f: Point → num

class Pointe {public int x,y} ; $\quad$ f(new Pointe(5, 7)) ?

JAVA + C++ use *nominal* subtyping

Go, Ocaml, Haskell use *structural* subtyping (so-called "duck" typing)

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$T ::= \ldots \mid \forall \cancel{A}. T \mid \forall A \leq \overset{F}{T}.\ T'$

Java: class $\overset{\text{variable}}{\text{BinaryTree}}$ < $\overset{A}{X}$ implements $\overset{T}{\text{Ordered}}$> $\overset{T'}{\{\ldots\}}$

F-bound Polymorphism

$$\frac{\Gamma \vdash M : \forall A \leq F.\ T' \qquad T \leq F}{\Gamma \vdash M[T] : T'[A \leftarrow T]}$$

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Dependent Types: $T ::= \ldots \mid \cancel{T \to T} \mid (x{:}T) \to T \mid (P\ x)$

quicksort : ($l$ : List num) → (sorted version of $l$) $\quad$ Coq, HOL, Isabelle, Agda, Idris,