22-1

Goal:

$\forall P.$ If $A(P)$ then P will not get stuck

if not iff because on incompleteness

"not get stuck" := run forever or reach a value

$(\forall P'. P \to^* P', \exists P''. P' \to P'')$ $(\exists V. P \to^* V)$

$(\exists P'.$

Progress: $\forall P.\ A(P) \Rightarrow$ ~~$\forall P'$~~ $P \to P')$ ~~[illegible]~~ $\vee$ P is a V

Preservation: $\forall P.\ A(P) \wedge \exists P', P \to P' \Rightarrow A(P')$

An example "A" is a type system that categorizes values and predicts the value ultimately returned (if one is)

$T := Num \mid Bool \mid T \to T$

$E := b \mid o^n\ E \ldots \mid X \mid \lambda X. E \mid E\ E$

$\Gamma := X \mapsto T$

Algorithm is "Is E in the type relation with some type?"

"$\Gamma \vdash E : T$" := "Gamma proves that E has type T"

$$\frac{}{\Gamma \vdash true : bool} \qquad \frac{}{\Gamma \vdash false : bool} \qquad \frac{}{\Gamma \vdash n : Num}$$

$$\frac{\Gamma \vdash E : num}{\Gamma \vdash iszero\ E : bool} \qquad \frac{\Gamma \vdash E_1 : Num \quad \Gamma \vdash E_2 : Num}{\Gamma \vdash E_1 + E_2 : Num} \qquad \frac{(X \mapsto T) \in \Gamma}{\Gamma \vdash X : T}$$

$$\frac{\Gamma[X \mapsto T_D] \vdash E : T_R}{\Gamma \vdash (\lambda X. E) : T_D \to T_R} \qquad \frac{\Gamma \vdash E_f : T_D \to T_R \quad \Gamma \vdash E_a : T_D}{\Gamma \vdash E_f\ E_a : T_R}$$

$E \ / \ (\lambda X. E) \ \# \ (\lambda X : T.\ E)$

$$\frac{\Gamma[X \mapsto T_B] \vdash E : T_R}{\Gamma \vdash (\lambda X : T_D.\ E) : T_D \to T_R}$$

22-2 / Conditionals

$E = \ldots \mid \text{if } E \; E \; E$

$$\frac{\Gamma \vdash E_c : \text{bool} \qquad \Gamma \vdash E_t : T_T \qquad \Gamma \vdash E_f : T_F}{\Gamma \vdash \text{if } E_c \; E_t \; E_f \vdash}$$

	option 1	option 2	option 3
premise	$T_T = T_F$		
conclusion	T_T	$T_T \cap T_F$	$T_T \cup T_F$

$$\frac{\Gamma \vdash E_c : \text{bool} \,/\, \Gamma_t \,/\, \Gamma_f \qquad \Gamma_t \vdash E_t : T_T \,/\, \Gamma_{tt} \,/\, \Gamma_{tf} \qquad \Gamma_f \vdash E_f : T_F \,/\, \Gamma_{ft} \,/\, \Gamma_{ff}}{\Gamma \vdash \text{if } E_c \, E_t \, E_f \; \cancel{\vdash} \; T_T \cup T_F \,/\, \Gamma_{tt} \cap \Gamma_{ft} \,/\, \Gamma_{tf} \cap \Gamma_{ff}}$$

nus

$[x \mapsto (\text{num} \cup \text{str})] \vdash$ (if (number? x)

(+ x 5)) $\to$ n

str

(string->number x)) : num

occurrence typing

$\Omega : T$? $\Omega = \omega \; \omega$ $\omega = \lambda x. \, x \; x$

$$\frac{\dfrac{\dfrac{\varnothing'' \vdash x : T_D \to T_R \qquad \varnothing'' \vdash x : T_D}{\varnothing[x \mapsto T_D] \vdash (x \; x) : T_R}}{\varnothing \vdash \omega : T_D \to T_R} \qquad \dfrac{\varnothing[x \mapsto T_D'] : x \; x : T_R'}{\varnothing \vdash \omega : T_D}}{\varnothing \vdash \omega \; \omega : T_R}$$

$T_D = T_D' \to T_R'$

$T_D = T_D \to T_R$

$(T_D' \to T_R') = (T_D' \to T_R') \to T_R$

No T_D exists!

$T_R' = T_R$

$T_D' = T_D' \to T_R'$

Strong Normalization — all programs have values

$M = \ldots \mid \text{fix } M$

$$\frac{\Gamma \vdash M : (T_D \to T_R) \to (T_D \to T_R)}{\Gamma \vdash \text{fix } M : T_D \to T_R}$$

$E[\text{fix } (\lambda X{:}T. \, M)] \mapsto E[M[X \leftarrow (\text{fix } (\lambda X{:}T. \, M))]]$

$T. \vdash \text{fix } (\lambda X{:}T. X) : T$

22-3

Pairs

$M ::= \dots \mid \text{pair } M_1\ M_2 \mid \text{fst } M_3 \mid \text{snd } M_2$

$T ::= (T \times T) \mid \dots$

$$\frac{\Gamma \vdash M_1 : T_1 \quad \Gamma \vdash M_2 : T_2}{\Gamma \vdash \text{pair } M_1\ M_2 : T_1 \times T_2} \qquad \frac{\Gamma \vdash M : T_1 \times T_2}{\Gamma \vdash \text{fst } M : T_1} \qquad \frac{\Gamma \vdash M : T_1 \times T_2}{\Gamma \vdash \text{snd } M : T_2}$$

Variants

$M ::= \dots \mid \text{inl } M \mid \text{inr } M \mid \text{case } M \text{ with } (\lambda X. M) \text{ or } (\lambda X. M)$

$T ::= \dots \mid (T + T)$

$$\frac{\Gamma \vdash M : T_1}{\Gamma \vdash \text{inl } M : T_1 + T_2} \qquad \frac{\Gamma \vdash M : T_2}{\Gamma \vdash \text{inr } M : T_1 + T_2} \qquad \frac{\Gamma \vdash M_1 : T_1 + T_2 \quad \Gamma[X \mapsto T_1] \vdash M_2 : T_R \quad \Gamma[Y \mapsto T_2] \vdash M_3 : T_R}{\Gamma \vdash \text{case } M_1 \text{ with } (\lambda X. M_2) \text{ or } (\lambda Y. M_3) : T_R}$$

$$\text{case } (\text{inl } V) \text{ with } (\lambda X. M_1) \text{ or } (\lambda Y. M_2) \longrightarrow M_1[X \leftarrow V]$$

Unit Type

$M ::= \dots \mid \text{unit}$ $\qquad T ::= \dots \mid 1$

$\Gamma \vdash \text{unit} : 1$ $\qquad \text{bool} ::= 1 + 1$

Null Type

$M ::= \dots$ $\qquad T ::= \dots \mid \emptyset$

$T ::= \emptyset \mid 1 \mid T + T \mid T \times T$ (algebraic data types)

$\text{bool} ::= 1 + 1$ $\qquad \text{binn} ::= 1 + \text{binn} + \text{binn}$ $\qquad \emptyset + 1 = 1$

$\text{nat} ::= 1 + \text{nat}$ $\qquad 1 \times Y = Y$

$\text{list } X ::= 1 + (X \times \text{list } X)$ $\qquad \emptyset \times Y = \emptyset$

$\text{bin } Y ::= 1 + (\text{bin } Y \times Y \times \text{bin } Y)$

$$\Delta_X \text{list } X = \emptyset + (X \times \delta_X \text{list } X) + (\overbrace{\delta_X X}^{=1} \times \text{list } X) \quad \text{(Zipper)}$$