$X \simeq \mathbb{N} \Rightarrow X$ is countable

Evens is countable:

must find function f that is one-to-one and onto.

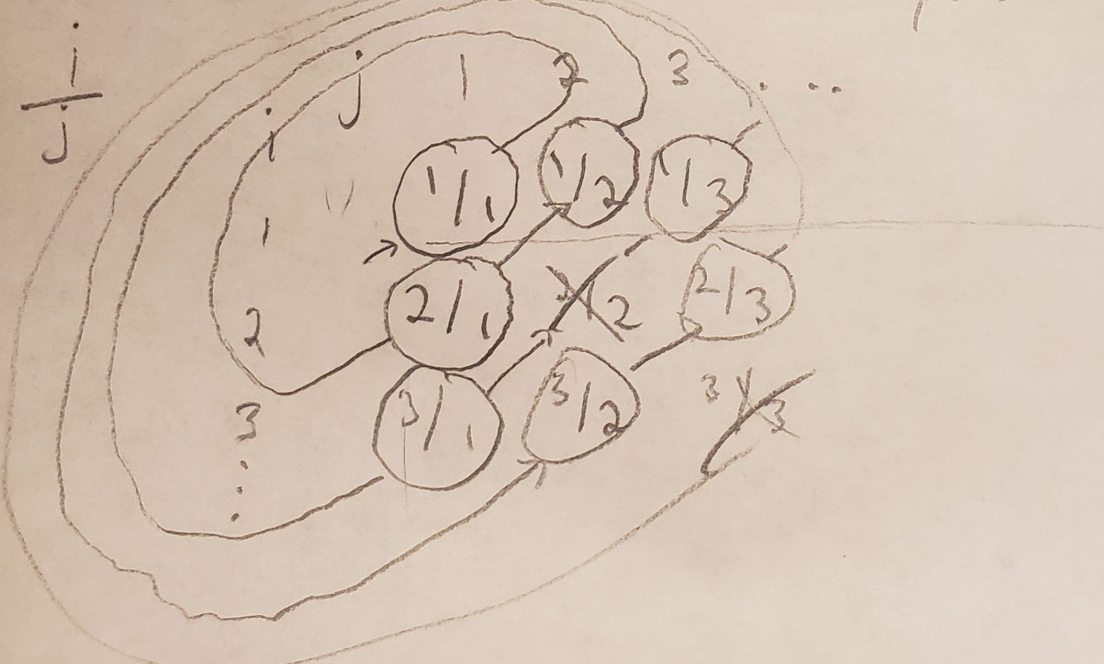
bijection or correspondence

$f(n) = 2n$

$\mathbb{N}$	$\xrightarrow{f}$ Even
1	2
2	4
3	6
4	8
⋮	⋮

$Q = \{ \frac{m}{n} \mid m, n \in \mathbb{N} \}$ = rational numbers

Is Q countable? yes.



Is $\mathbb{R}$ countable? No.

Prove by contradiction w/ diagonalization method.

imagine some bijection f

$\mathbb{N} \rightarrow f \rightarrow \mathbb{R}$

$\exists x \in \mathbb{R}.\ \forall n \in \mathbb{N}.\ f(n) \neq x$

1	3.14159...
2	55.555...
3	0.7549...
⋮	

Choose x: nth digit of the decimal place of x is not equal to the nth digit of f(n).

0.367...

f is $\neg$onto $\Rightarrow$ $\mathbb{R}$ are bigger than $\mathbb{N}$

$\mathbb{R}$ is uncountable.

Is IBS countable? No.

imagine f that did map.

$\mathbb{N}$	IBS
1	0 0 0 0 0 ....
2	1 1 1 1 1 ....
3	0 1 0 1 0 ....
4	1 0 1 0 1 ....
5	1 1 0 0 1 ....
⋮	⋱

x=10110...

Countable	Uncountable
N	Reals
||	||
N x N (Q)	IBS
||	||
TM	ALL

Not empty

TM is countable

must first show that Σ^* is countable

$\Sigma = \{0,1\}$

$\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 11, 000, 001, 011 \ldots\}$

"Lexicographic ordering"

= strings of length i appear before strings of length j whenever $i<j$

and within length i, strings with lower binary values come first.

$\Sigma^* \simeq N$ (is countable) lexicographic ordering is a bijection

$f(0) = \varepsilon$

$f(1) = 0$

$\vdots$ $f(4) = 01$

$\langle M \rangle$

TM is countable because each TM M has an encoding into a string $\langle M \rangle$

$\langle M \rangle \in \Sigma^*$

Just omitting strings from Σ^* that are not TM.

ALL is uncountable

IBS is uncountable. ALL $= P(\Sigma^*)$

Show ALL is uncountable by giving correspondence with IBS.

f: ALL $\rightarrow$ IBS

$\Sigma^* = \{s_1, s_2, s_3 \ldots$

Characteristic sequence. $\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000\}$

$f(A)[i] = 1 \mid s_i \in A$ $A = \varepsilon$ 0, 00, $000\}$

$0 \mid s_i \notin A$ $f(A) = 0\,1\,0\,1\,1\,0\,0\,1$

A is all words that start with 0

ALL languages is uncountable.

An Undecidable language

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a turing machine and } M \text{ accepts } w\}.$$

Proof by contradiction.

Suppose H is a decider for A_{TM}

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ rejects } w \end{cases}$$

New TM D that calls H as subroutine

D = "On input $\langle M\rangle$, where M is a TM:

1. Run H on input $\langle M, \langle M\rangle\rangle$.
2. Output the opposite of what H outputs.

$$D(\langle M\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ does Not accept } \langle M\rangle \\ \text{reject} & \text{if } M \text{ does accept } \langle M\rangle \end{cases}$$

Now run D with its own description as input.

$$D(\langle D\rangle) = \begin{cases} \text{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \text{reject} & \text{if } D \text{ does accept } \langle D\rangle \end{cases}$$

H accepts $\langle M, w\rangle$ exactly when M accepts w.

D rejects $\langle M\rangle$ exactly when M accepts $\langle M\rangle$

D rejects $\langle D\rangle$ exactly when D accepts $\langle D\rangle$

if $D(\langle D\rangle) = acc$, then $D(\langle D\rangle)$ must $\neq acc$.

Liar's paradox "This statement is false"

QED. A_{TM} is undecidable.