20-1

$A_X = \{ \langle M, w \rangle \mid M \text{ is an } X \text{ and } w \in L(M) \}$

$A_{DFA} \in \Sigma_0$ $\quad$ $A_{CFG} \in \Sigma_0$

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } w \in L(M) \}$

- Turing Machine that interpreters other Turing Machines
- MAME

$A_{TM} \in \Sigma_1$: Input : $\langle M, w \rangle$

	Tape 1: $\langle M \rangle$	$\langle M \rangle$	
U	Tape 2: $\langle [\,]w \rangle$	$\Rightarrow \langle u_i [\,] w_i \rangle$	$[q_0]w \Rightarrow u_i [q_i] w_i$
= the universal Turing Machine	Tape 3: $\langle q_0 \rangle$	$\langle q_i \rangle$	in M
	~~Tape~~		

Suppose M_A is an acceptor (not decider) then what does $U(\langle M_A, w \rangle)$ where M_A diverges on w do?

$U \in \Sigma_1$: Reject / Diverge $\qquad$ $U \in \Sigma_0$: Reject / ~~Diverge~~

```
while (simulate machine not in accept or reject) {
    simulate 1 step();
    if (machine is diverging) {break; }
}
return acept or reject;
```

20-2

When are two sets the same size?

$\simeq$:= same size

$A \simeq B \iff C(A) = C(B)$

"Count them and compare numbers"

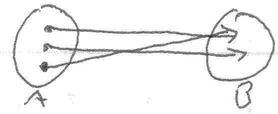
Naturals $\simeq 0^*1$

C: Finite Set $\to N$

$\not$ Set $\to N$

$A \simeq B \iff$ build a mapping

A mapping is a function from A to B



$f(x) = x^2$ is a fun but not a mapping

+ one-to-one (don't double-count Bs)

$(\forall a_1, a_2 . \; f(a_1) = f(a_2) \Rightarrow a_1 = a_2)$

+ onto (don't leave out Bs)

$(\forall b . \exists a . \; f(a) = b)$

Naturals $= \{0, 1, 2, 3, 4, \ldots\}$

Evens $= \{0, 2, 4, 6, 8, \ldots\}$

positive: $(\exists f : N \to E . \; (\forall a_1, a_2 . \; f(a_1) = f(a_2) \Rightarrow a_1 = a_2)$
$\wedge (\forall b . \exists a . \; f(a) = b))$

negative: $(\forall f : N \to E . \; (\exists a_1, a_2 . \; f(a_1) = f(a_2) \Rightarrow a_1 \neq a_2)$
$\vee (\exists b . \forall a . \; f(a) \neq b))$

$f(x) = 2 \cdot x$ ✓ $2a_1 = 2a_2 \Rightarrow a_1 = a_2$

$2 \cdot x = 2 \cdot b \Rightarrow x = b$

If $X \simeq N$, then X is countable

(N: naturals)

X is enumerable

20-3

Naturals and the Rationals

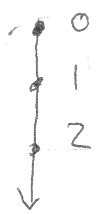
$\mathbb{N}$ $\mathbb{Q}$ x/y

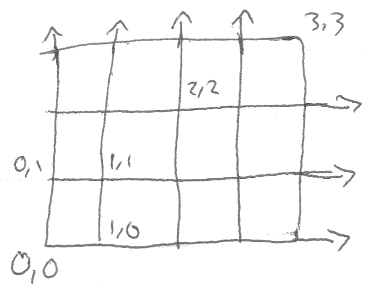
3 $3/1$ $3/2$ $3/3$ $3/4$ ...

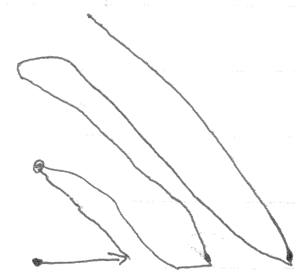
$1/3$ $2/3$ $3/3$ $4/3$ ...

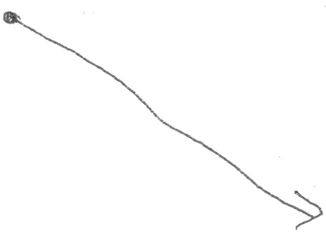
4 $4/$ $4/$ $4/$

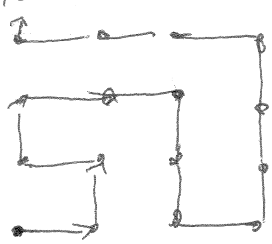
$/4$ $/4$ $/4$

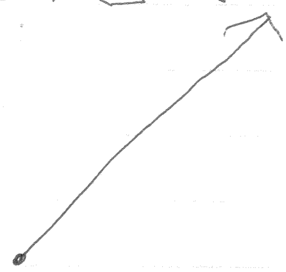












$\mathbb{N} \simeq \mathbb{N} \times \mathbb{N}$

$\mathbb{N}^2$

$M(x,y) = x/y = \mathbb{Q}$

$M(x,y) = x-y = \mathbb{Z}$

$M(x,y) = (x,y) = \text{Plane}$

$$f(x,y) = \frac{(x+y)(x+y+1)}{2}$$

Cantor's Pairing Function

$\mathbb{N}^* \simeq \mathbb{N}^k$

$\underbrace{\underbrace{\mathbb{N} \times \mathbb{N}}_{\mathbb{N}} \times \mathbb{N}}_{\mathbb{N}}$ $(k=3)$

20-4

Naturals and Reals

$\mathbb{N}$ $\mathbb{R}$

$0 \in \mathbb{R}$ $1 \in \mathbb{R}$

$0.5 \in \mathbb{R}$

$\pi \in \mathbb{R}$ $e \in \mathbb{R}$

$k\pi \in \mathbb{R}$, $k \in \mathbb{N}$

"Dedekind cut"
= Two infinite sets of $\mathbb{Q}$: (u, l) u's are bigger than r, and l's are smaller

"Cauchy Sequence"
= an infinite seq of rationals converging to r

A real is a function from $\mathbb{N}$ to a digit

$R_{01} = [0, 1)$ = binary encoded

$R_{01} \equiv \mathbb{N} \to \{0, 1\}$ 0.0000 . . . .

"0" = $\lambda n.\ 0$

$\lambda n.$ if $n \% 2 = 0$, 1
o.w. 0
= "$0.\overline{10}$"

"0.5" = $\lambda n.$ if $n == 0$, 1
o.w., 0

0.10000...

"π" = $\lambda n.$ compute π to n, then return that digit

N	R		
0	π	$= \lambda$	= 0.010111 ...
1	$.\overline{3}$	$= \lambda$	= 0.101100 ...
2	.5	$= \lambda$	= 0.10111 ...
3	.45	$= \lambda$	= 0.01100 ...

$f(n) = \lambda pos.$ if $pos = n$, 1
o.w. 0

$0 \to .1\overline{0}$

$1 \to .01\overline{0}$

$2 \to .001\overline{0}$

$\forall m : \mathbb{N} \to \mathbb{R}.$

$\exists b \in \mathbb{R}.\ \forall a \in \mathbb{N}.\ f(a) \neq b$

choose b: $(\lambda pos. \neg(m(pos))(pos))$

$\mathbb{N} \not\cong \mathbb{R}$

$\neg$onto $\Rightarrow$ R are bigger

Lindsey please review this part