3-1

$N, M ::= X \mid M\ N \mid \lambda X. M$

unbound and must be guessed

$$\frac{\Gamma[X \to D] \vdash M : R}{\Gamma \vdash \lambda X. M : D \to R}$$

$N, M ::= X \mid M\ N \mid \lambda X{:}T.\ M$

$$\frac{\Gamma[X \mapsto D] \vdash M : R}{\Gamma \vdash \lambda X{:}D.\ M : D \to R}$$

Type Inference — "$\lambda X. M$" $\to$ "$\lambda X : T. M$"

infers the type

$\lambda X.\ X+2$ X must a num   guarantee that

$(\lambda X.\ X)\ 2$ X must a num   type check will pass

$(\lambda X.\ X+2)$ "two"   has no type

Type inference = constraint generation + constraint solving

"$X = \text{num}, X = \text{str}$"   "false"

"$X = \text{num}, Y = X$"   "$Y = \text{num}$"

$C = T = T \qquad T = B \mid T \to T \mid \cancel{A}$

$\Gamma \vdash M : T\ ;\ P(C)\ ;\ P(A)$ $\Gamma \vdash M : T$

$$\frac{\Gamma[X \to D] \vdash M : R\ ;\ C_1\ ;\ A_1}{\Gamma \vdash \lambda X. M : D \to R\ ;\ C_1\ ;\ A_1 \cup \{D\}}\ \text{lam}$$

$$\frac{\Gamma(X) = T}{\Gamma \vdash X : T\ ;\ \emptyset\ ;\ \emptyset}\ \text{var}$$

$$\frac{}{\Gamma \vdash b : B(b)\ ;\ \emptyset\ ;\ \emptyset}\ \text{con}$$

$$\frac{\Gamma \vdash M : T_1\ ;\ C_1\ ;\ A_1 \qquad \Gamma \vdash N : D\ ;\ C_2\ ;\ A_2}{\Gamma \vdash M\ N : R\ ;\ C_1 \cup C_2 \cup \{T_1 = D \to R\}\ ;\ A_1 \cup A_2 \cup \{R\}}\ \text{app}$$

app: $\emptyset \vdash (\lambda X. X)\ 5 : R\ ;\ \{D \to D = \text{Num} \to R\}\ ;\ \{D, R\}$ $D \to D = \text{Num} \to R$

lam: $\emptyset \vdash \lambda X. X : D \to D\ ;\ \emptyset\ ;\ \{D\}$ $\emptyset \vdash 5 : \text{Num}\ ;\ \emptyset\ ;\ \emptyset$

var: $X \to D \vdash X : D\ ;\ \emptyset\ ;\ \emptyset$

$\Gamma(X) = D$

23-2/ constraint solving

$x+2=18$, what is x? $x=16$

$x+y=16$
$x+3y=8 \Rightarrow 2y=-8$

$x+2=18$ what is x? Nothing?
$x+3=20$

$$\begin{bmatrix}1&1&16\\1&3&8\end{bmatrix} \rightarrow \begin{bmatrix}1&1&16\\0&2&-8\end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix}1&0&20\\0&1&-4\end{bmatrix} \leftarrow \begin{bmatrix}1&1&16\\0&1&-4\end{bmatrix}$$

$x+2=18$ what is x & y? $x=16$
$y+3=x$ $y=13$

Gaussian Elimination (n^3)

$x+y=16$
$2x+2y=32$

Unification = G.E. for arbitrary data

$U : P(C) \times P(A=T) \rightarrow P(A=T) + \text{Error}$

constraints, solutions, final solution

$U(\emptyset, S) = S$

$U(\{T=T\} \cup C, S) = U(C, S)$

$U(\{T_1 \rightarrow T_2 = T_3 \rightarrow T_4\} \cup C, S) = U(\{T_1 = T_3, T_2 = T_4\} \cup C, S)$

$U(\{A=T\} \cup C, S) = U(C[A \leftarrow T], S[A \leftarrow T] \cup \{A=T\})$

$U(\{T=A\} \cup C, S) = U(\{A=T\} \cup C, S)$ if T contains A, then error

$U(\{B = T_1 \rightarrow T_2\} \cup C, S) = \text{error}$

$U(\{B_1 = B_2\} \cup C, S)$ where $B_1 \neq B_2$ = error

$T = \dots \mid T \times T$ $U(\{T_1 \times T_2 = T_3 \times T_4\} \cup C, S) = U(\{T_1 = T_3, T_2 = T_4\} \cup C, S)$

$\Gamma \vdash M : X ; C_1 ; A_1$

$\Gamma \vdash \text{pair } M\ N : T_1 \times T_2 ; C_1 \cup C_2 ; A_1 \cup A_2$ when $\Gamma \vdash M : T_1 ; C_1 ; A_1$

$\Gamma \vdash N : T_2 ; C_2 ; A_2$

$\Gamma \vdash \text{fst } M : T_1 ; C_1 \cup \{X = T_1 \times T_2\}, A_1 \cup \{T_1, T_2\}$

$U(\{D \rightarrow D = Num \rightarrow R\}, \emptyset) = U(\{D = Num, D = R\}, \emptyset) = U(\{Num = R\}, \{D = Num\})$
$= U(\{R = Num\}, \{D = Num\}) = U(\emptyset, \{D = Num, R = Num\})$

3-3

$$((\underbrace{\lambda f.\ f\ 0}_{f:A \quad A\to B})\ \underbrace{(\lambda x. x)}_{C\to C}) : D$$

$U(\{A = num \to B,\ A\to B = (C\to C)\to D\},\ \emptyset)$

$= U(\{(num\to B)\to B = (C\to C)\to D\},\ \{A = num\to B\})$

$= U(\{(num\to B) = (C\to C),\ B = D\},\ \text{''})$

$= U(\{num = C,\ C = B,\ B = D\},\ \text{''})$

$= U(\{num = B,\ B = D\},\ \{A = num\to B,\ C = num\})$

$= U(\{num = D\},\ \{A = num\to num,\ C = num,\ B = num\})$

$= U(\emptyset,\ \cup \{D = num\})$

$= ans$

under constrainted = $\{\}$ = polymorphism

over constraind = (num = bool) = no type

$$\frac{\emptyset \vdash (\lambda x. x) : D\to D;\ \emptyset;\ \{D\}}{\begin{array}{l}\emptyset[x\to D] \vdash x : D;\ \emptyset;\ \emptyset \\ \Gamma(x) = D\end{array}}$$

$(\lambda x: x) : (D\to D)$

$\forall D.\ D\to D$

principal type = most polymorphic type possible

$U(\{A = A\to A,\ A\to A = A\},\ \emptyset)$

$U(\ ,\ (A\to A)\to(A\to A) = (A\to A),\ \{A = A\to A\})$

$U(A \Rightarrow A = A,\ A\to A = A,\ \text{''})$

$\emptyset \vdash \Omega : A;\ ;\ \{A\}$ "occurs problem" "A occurs in T"

$((\lambda i.$

$\quad (\text{if } (i\ true)\ (i\ 5)\ (i\ 6)))$

$(\lambda x. x))$

let-polymorphism

let $i = \lambda x. x$ in

if $(i\ true)\ (i\ 5)\ (i\ 6)$

$$\frac{\Gamma \vdash N[x\leftarrow M] : T}{\Gamma \vdash \text{let } x = M \text{ in } N : T}$$