4-3

$eval_v : M \longrightarrow$ answers $= A$

$$eval_v(M) = \begin{cases} b & \text{if } M =_v b \\ \text{'fun} & \text{if } M =_v \lambda X.N \end{cases}$$

$\cancel{X}$

answers $= V$ (constant, function, or variable)

b $(\lambda X. M) \to$ 'function $X \to X \in FV(M)$

$A = B \cup \{\text{'fun}\}$

Is $eval_v$ total or partial?

partial:

$$\left\{\begin{array}{l} M =_v X \\ M =_v (b\ M) \\ \quad =_v (o^n\ b \ldots (\lambda X.N)\ b' \ldots) \end{array}\right.$$

"stuck" (code for "error")

never finish = "diverge" (infinite loop)

$\Omega = ((\lambda x.(x\ x))\ \underline{(\lambda y.(y\ y))})$ $M\ \beta_v\ M$

β_v

$((\lambda y.(y\ y))\ (\lambda y.(y\ y)))$

Y = the Y combinator (or fixed-point operator)

$$Y_v = (\lambda f.\ (\lambda x.\ ((\lambda g.\ (f\ (\lambda x.\ ((g\ g)\ x))))$$
$$(\lambda g.\ (f\ (\lambda x.\ ((g\ g)\ x))))\ x)))$$

Theorem: If $K = \lambda Z.\lambda X.L$ then $(K\ (Y_v\ K)) =_v (Y_v\ K)$

$Y_v\ K \Rightarrow_v \ldots [f \leftarrow K] = V$

$Y_v\ K \Rightarrow V \rightarrow_v \lambda x.((K\ V)\ x)$

$= \lambda x.(([\lambda Z.\lambda X.L]\ V)\ x)$

$\twoheadrightarrow_v \lambda x.\ L[Z \leftarrow V][X \leftarrow x]$

$= \lambda X.\ L[Z \leftarrow V]$

$\leftarrow_v ((\lambda Z.\lambda X.L)\ V)$

$= (K\ V)$

$\leftarrow_v (K\ (Y_v\ K))$

4-4 If $M =_v V$, $\forall X, N$,

$$((\lambda X. N)\ M) =_v N[X \leftarrow M] \quad (\beta)$$

justifies inlining (pg 51)

5-1 $eval_v$ is a partial function

$(+ \ (+\ 1\ 2) \ (+\ 3\ 4))$

$\swarrow \qquad \searrow$

$(+\ 3\ (+\ 3\ 4)) \qquad (+\ (+\ 1\ 2)\ 7)$

(++i / i++)

$\forall M, A_1, A_2.\ eval_v(M) = A_1 \wedge eval_v(M) = A_2 \rightarrow$

$A_1 = A_2 \qquad =_v$

A_1 \ A_2	b	fun
b	trivial	•
fun	•	•

$\lambda X. M =_v b$

If $M =_v N$, $\exists L.\ M \twoheadrightarrow_v L$ and $N \twoheadrightarrow_v L$.

$$\begin{tikzcd} M \arrow[dr, ->>] & & N \arrow[dl, ->>] \\ & L & \end{tikzcd}$$

Consistency

Church-Rosser

If $L \twoheadrightarrow_v M$ and $L \twoheadrightarrow_v N$, $\exists K$.

$M \twoheadrightarrow_v K$ and $N \twoheadrightarrow_v K$

$$\begin{tikzcd} & L \arrow[dl, ->>] \arrow[dr, ->>] & \\ M \arrow[dr, ->>] & & N \arrow[dl, ->>] \\ & K & \end{tikzcd}$$

B - C.R. was true for $\rightarrow_r$

S-2

$((\lambda x. (x\ x))\ (\lambda y. ((\lambda x.x)(\lambda x.x))))$

$((\lambda y. ((\lambda x.x)(\lambda x.x)))\ (\lambda y. ((\lambda x.x)(\lambda x.x))))$

$\hookrightarrow_v ((\lambda x. (x\ x))\ (\lambda y. (\lambda x.x)))$

$\to ((\lambda y. (\lambda x.x))\ (\lambda y. (\lambda x.x)))$

$M \hookrightarrow_v M$

$A: \quad (o^n\ b_1 \ldots b_n) \hookrightarrow \delta(o^n, b_1, \ldots, b_n)$

$\beta_v' : \quad ((\lambda X. M)\ U) \hookrightarrow_v M'[X \leftarrow V]$

if $M \hookrightarrow_v M'$ and $U \hookrightarrow_v V'$

$(M\ N) \hookrightarrow_v (M'\ N')$ if $M \hookrightarrow_v M'$ and $N \hookrightarrow_v N'$

$(\lambda X. M) \hookrightarrow_v (\lambda X. M')$

$(o^n\ M_1 \ldots M_n) \hookrightarrow_v (o^n\ M_1' \ldots M_n')$

Substitution Lemma:

If $M \hookrightarrow_v M'$ and $N \hookrightarrow_v N'$, then

$\underline{M[X \leftarrow N]} \hookrightarrow_v M'[X \leftarrow N']$

$M = (+\ (+\ 3\ 10)\ (x\ 5))$ $\quad M' = (+\ 13\ (x\ 5))$

$N = (\lambda y. (+\ \cancel{}\ (+\ 2\ 4)\ y))$

$N' = (\lambda y. (+\ 6\ y))$

$(+\ (+\ 3\ 10)$
$\quad ((\lambda y. (+\ (+\ 2\ 4)\ y))\ 5))$

$(+\ 13\ ((\lambda y. (+\ 6\ y))\ 5))$

5-3 Observational Equivalence

What does $=_v$ mean?

↳ sym (refl + trans (compatible ($v = \Delta \cup \beta_v$)))

Want : do these programs do the same thing?

slow fast	ver1 ver2	platform1 platform2
proof, use fast	check	porting
more clear, tricks		

$(+ \; 3 \; 4) =_v 7$

FR ($\lambda x.$ ... factorial recursive)

$=_v$? IF($\lambda x.$ ... iterative factorial ...)

X no!

$\Omega = ((\lambda x. (x x)) (\lambda x. (x \; x)))$

$(\lambda x. \Omega)$ diverges $\neq_v$ $(\lambda x. (\Omega \; \Omega))$ diverges

$=_J$: turn the program M into a math function

↳ denotational semantics (Scott Domain Theory)

① $\exists x. \; (FR \; x) \neq (IF \; x)$

isort vs qsort

② (a a → bool) (L a) → (L a)

$\exists$ cmp, l, isort(cmp, l) = qsort(comp, l)

F and G

③ $X \to (Y \to Z)$ $\quad F x = G x$

$\exists x, \exists y \;\; ((F \; x) \; y) \neq ((G \; x) \; y)$

Context is a program with a HOLE

① C = ([] x) $\qquad$ ([] 7)

③ C = (([] x) y) $\qquad$ (([] 10) 1)

5-4

$$
\begin{aligned}
C = & \; [\,] \\
| & \; (\lambda X.\ C) \\
| & \; (C\ M) \\
| & \; (M\ C) \\
| & \; (o^n\ M \ldots C\ M \ldots)
\end{aligned}
$$

(There's nothing for X or b)

$C[N]$ to mean "fill the hole in C with N"

$[\,][N] = N$

$(\lambda X.\ C)[N] = (\lambda X.\ C[N])$ // N <u>can</u> mention X

$(C\ M)[N] = (C[N]\ M)$

$(M\ C)[N] = (M\ C[N])$

$(o^n\ M \ldots C\ M \ldots)[N] = (o^n\ M \ldots C[N]\ M \ldots)$

$C = \lambda X.[\,]$ $\quad N = x \quad C[N] = \lambda x.x$

$\neq \lambda x.y \quad \neq \lambda y.x$

$M \rightarrow_{cv} N$ iff

$\exists C.\ M = C[M']$ and $N = C[N']$ and $M'\ v\ N'$

$\rightarrow_{cv}\ \neq\ \rightarrow_v$ $\quad M = (x\ (+\ 3\ 4)) \quad N = (x\ 7)$

$C = (x\ [\,]) \quad M' = \cancel{(+34)}\ (+\ 3\ 4) \quad N' = 7$

$M \simeq_v N$ (M is observationally equivalent to N)

iff. $\forall C.\ \text{eval}(C[M]) = \text{eval}_v(C[N])$

wants: $\forall m.\ M \simeq_v M$

if $L \simeq_v M$ and $M \simeq_v N$, $L \simeq_v N$ (equivalence)

$L \simeq_v M \Rightarrow M \simeq_v L$

if $M \simeq_v N$, $\forall C.\ C[M] \simeq_v C[N]$

5-5 Soundness + Incompleteness

If $M =_v N$, then $M \simeq_v N$. (soundness)

But, $M \simeq_v N \not\Rightarrow$ that $M =_v N$ (incompleteness)

$\swarrow$

$(\lambda x.\, \Omega) \simeq_v (\lambda x.\, (\Omega\ \Omega))$ $\qquad =_v \subsetneq \simeq_v$

but

$(\lambda x.\, \Omega) \neq_v (\lambda x.\, (\Omega\ \Omega))$

$(\Omega\ \Omega)$ $\qquad$ (loop())(7) $\simeq$ loop()

$\downarrow$

η (eta) : $(\lambda x.\, Mx) \Rightarrow M$

$(\lambda x.\, \Omega x)$ $\qquad \Omega$

$C = [\,]$ $\quad \downarrow$ fun $\qquad \downarrow$ diverges