21-1

$\Gamma \vdash e : T$

Gamma proves programe has type T

type judgment

$T = \text{num} \mid \text{bool} \mid (T \to T)$

$$\frac{\Gamma[x \mapsto T] \vdash M : T'}{\Gamma \vdash (\lambda x : T.\, M) : (T \to T')} \qquad \frac{\Gamma \vdash f : (T \to T') \quad \Gamma \vdash a : T}{\Gamma \vdash (f\ a) : T'}$$

Soundness : If $\vdash M : T$, then $M \to\!\!\!\to_v V$ or
$M \to\!\!\!\to_v M'$ and $M' \mapsto\!\!\!\to_v M''$ and $V, M', M'' : T$

Preservation : If $M \to_v N$, and $\vdash M : T$, then $\vdash N : T$

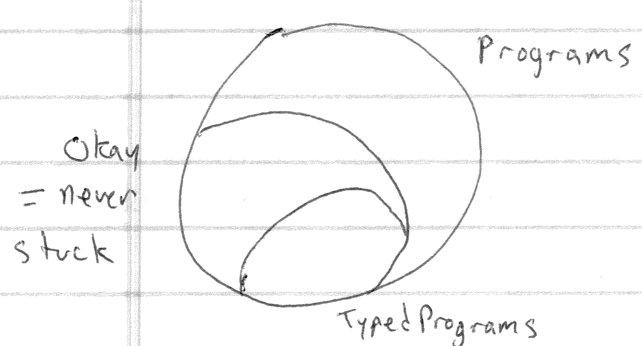
Progress : If $\vdash M : T$, then either $M \in V$ or $M \to_v N$

$\Lambda \quad e = x \mid e\ e \mid \lambda x.e \qquad E[(\lambda x.e)\ e'] \to E[e[x \leftarrow e']]$

$\ulcorner 0 \urcorner = \lambda s.\lambda z.\ z \qquad \ulcorner \text{true} \urcorner = \lambda t.\lambda f.\ t$

$\ulcorner 2 \urcorner = \lambda s.\lambda z.\ s\,s\,z \qquad \ulcorner \text{false} \urcorner = \lambda t.\lambda f.\ f$

$Y \to$ recursion $\qquad \Omega = \underbrace{(\lambda x.xx)}_{\omega}(\lambda x.xx)$



Soundness = Typed $\subseteq$ Okay

Incompleteness/Gödel = Typed $\neq$ Okay

$\boxed{\text{Strong Normalization}}$

$(\lambda x.(x\ x))$ untyped $\qquad (\lambda x : T.\ (x\ x))$ typed $\qquad \boxed{T_1 = T_1 \to T'}$

$$\frac{\dfrac{\dfrac{T = T_1 \to T'}{\Gamma[x \mapsto T] \vdash x : T_1 \to T'} \qquad \dfrac{T = T_1}{\Gamma[x \mapsto T] \vdash x : T_1}}{\Gamma[x \mapsto T] \vdash (x\ x) : T'}}{\Gamma \vdash (\lambda x : T.\ (x\ x)) : T \to T'}$$

$T_2' = T_2 \to T_3$

$(T_2 \to T_3) = (T_2 \to T_3) \to T'$

$T_1 = (T_2 \to T_3)$

21-2

$(\text{if0}\ k\ M\ N) \doteq (((\text{iszero}\ k)\ (\lambda X{:}\text{Num}.\ M)\ (\lambda X{:}\text{num}.\ N))\ 0)$

true $\quad x\ y \to x$

$(x\ \ 0) \to M$

$$M = \ldots \mid \text{true} \mid \text{false} \mid (\text{if}\ M\ M\ M)$$

$$T = \text{num} \mid \text{bool} \mid (T \to T)$$

$$E = \ldots \mid (\text{if}\ E\ M\ M)$$

$$E[(\text{if true}\ M\ N)] \to E[M]$$
$$E[(\text{if false}\ M\ N)] \to E[N]$$

$$\frac{\Gamma \vdash e_1 : \underbrace{\quad}_{}^{\text{bool}} \qquad \Gamma \vdash e_2 : \underbrace{\quad}^{T_2} \qquad \Gamma \vdash e_3 : \underbrace{\quad}^{T_3}}{\Gamma \vdash (\text{if}\ e_1\ e_2\ e_3) : \underbrace{\cancel{T_2 \cup T_3}}}$$ $\quad T_2 = T_3$

← like Typed Racket

((if (number? x) + string-append)

$\cancel{T_2 \cap T_3}$ ← hard to produce, no language does this

x $\qquad T_2$

```
(if (number? x)
    (+ x 1)
    "foo")    )
```

(x == 0 ? 0 : "foo")

21-3/

$m = \ldots$
$\quad \mid (fix\ m)$

$T = num \mid bool \mid T \to T$

$E = \ldots \mid (fix\ E)$

$$E[(fix\ (\lambda X{:}T.\ m))]$$
$$\longrightarrow\ E[m[X \longleftarrow (fix\ (\lambda X{:}T.\ m))]]$$

fixed-point of a fun F is a value x_0 s.t. $F\ x_0 = x_0$

$$\frac{\Gamma \vdash m \vdash ((T_1 \to T_2) \to (T_1 \to T_2))}{\Gamma \vdash (fix\ m) : (T_1 \to T_2)}$$

```
int f(int n){
  f(n);
}
```

$$((fix\ \overset{m}{(\lambda f{:}(num \to \overset{bool}{\cancel{num}}).\ \lambda n{:}num.\ (f\ n))})\ 0)$$

$E = ([\,]\ 0)\ \ [(fix\ m)]$

$\mapsto\ \ [\lambda n{:}num.\ ((fix\ m)\ n)]$

$E = [\,]\ \ [(\lambda n{:}num.\ (fix\ m)\ n)\ \ 0]$

$\mapsto\ \ [(fix\ m)\ \ 0]$

$$\boxed{\forall T.\quad \emptyset \vdash (fix\ (\lambda X{:}T.\ X)) : T}$$

2-1-4 / Data — Pairs

$M = \ldots$
$\quad | \; \text{pair } M \; M$
$\quad | \; \text{fst } M$
$\quad | \; \text{snd } M$

$V = (\text{pair } V \; V) \; | \ldots$

$E = \text{pair } E \; M$
$\quad \text{pair } V \; E$
$\quad \text{fst } E$
$\quad \text{snd } E$

$T = \ldots$
$\quad | \; (T \times T)$

$$E[(\text{fst } (\text{pair } V_1 \; V_2))] \longrightarrow E[V_1]$$
$$E[(\text{snd } (\text{pair } V_1 \; V_2))] \longrightarrow E[V_2]$$

$$\frac{\Gamma \vdash e_1 : (T_1 \times T_2)}{\Gamma \vdash (\text{fst } e_1) : T_1} \qquad \frac{\Gamma \vdash e_1 : (T_1 \times T_2)}{\Gamma \vdash (\text{snd } e_1) : T_2} \qquad \frac{\Gamma \vdash e_i : T_i}{\Gamma \vdash (\text{pair } e_1 \; e_2) : T_1 \times T_2}$$

Data – unions (dog or cat)

$M = \ldots$
$\quad | \; (\text{inL } M) \; | \; (\text{inR } M)$
$\quad | \; (\text{match } M \; (\lambda X. N_1) \; (\lambda X. N_2))$

$V = (\text{inL } V) \; | \; (\text{inR } V)$

$E = (\text{inL } E) \; | \; (\text{inR } E)$
$\quad (\text{match } E \; M \; M)$

$T = \ldots$
$\quad | \; (T + T)$

$$\frac{\Gamma \vdash e_1 : T_1}{\Gamma \vdash (\text{inL } e_1) : T_1 + T_2} \qquad \frac{\Gamma \vdash e_2 : T_2}{\Gamma \vdash (\text{inR } e_2) : T_1 + T_2}$$

$$\frac{\Gamma \vdash M : T_1 + T_2 \qquad \Gamma \vdash N_1 : T_1 \to T \qquad \Gamma \vdash N_2 : T_2 \to T}{\Gamma \vdash (\text{match } M \; N_1 \; N_2) : T}$$