$$((\lambda x.x)\ \underbrace{((\lambda y.y)\ 5)})$$

$\downarrow x$ $\qquad\qquad\qquad \searrow$

$((\lambda y.y)\ 5) \qquad\qquad ((\lambda x.x)\ 5)$

$\qquad\qquad\qquad\qquad\qquad \downarrow$

$\qquad \longrightarrow \qquad 5$

eval$_v$ – is partial (errors (stuck terms), divergence)

$(5\ 6)$

$$(\text{add}\ \ (\text{add}\ 1\ 2)\ \ \ (\text{add}\ 3\ 4))$$

$\swarrow \Delta \qquad\qquad \swarrow \Delta \qquad \Delta = (o^n\ V_0 \dots V_n)$

$(\text{add}\ 3\ (\text{add}\ 3\ 4)) \qquad (\text{add}\ (\text{add}\ 1\ 2)\ 7) \Rightarrow \delta(o^n, V_0, \dots, V_n)$

"M" $\qquad \searrow \qquad \swarrow \qquad$ "N"

$(\text{add}\ 3\ 7) = L$

$\downarrow$

10

Consistency:

If $M =_v N$, $\exists L.\ M \twoheadrightarrow_v L$ and $N \twoheadrightarrow_v L$

$$\Rightarrow \frac{M =_v N}{{}_v\!\searrow\!\!\searrow\ L\ \swarrow\!\!\swarrow_v}$$

Church-Rosser

If $K \twoheadrightarrow_v M$ and $K \twoheadrightarrow_v N$, then $\exists L.$
$M \twoheadrightarrow_v L$ and $N \twoheadrightarrow_v L$

~~Cont~~ compatible closure $(\longrightarrow_v)$

"find any work you can and do it"

$\hookrightarrow_v$: ISWIM $\to$ ISWIM

$(o^n\ V_1\ \ldots\ V_n) \hookrightarrow_v \delta(o^n, V_1, \ldots, V_n)$ [Δ]

$(\lambda x.\ M)\ V \hookrightarrow_v M[X \leftarrow V]$ [β_v]

$(M\ \ N) \hookrightarrow_v (M'\ \ N')$
if $M \hookrightarrow_v M'$
and $N \hookrightarrow_v N'$

$(\lambda X.\ M) \hookrightarrow_v (\lambda X.\ M')$
if $M \hookrightarrow_v M'$

$(o^n\ M_1\ \ldots\ M_n) \hookrightarrow_v (o^n\ M_i\ \ldots\ M_n')$
if $M_i \hookrightarrow_v M_i'$

(add (add 1 2) (add 3 4))

$\hookrightarrow_v$

(add 3 7)

$\hookrightarrow_v$

10

f(x, y, z)

x = 0

f(++x, ++x, ++x)

f(1, 2, 3)

f(2, 1, 3)

f(3, 1, 2)

```
int main() { return 42; }   ⟶ 42
int f() { f(); }
```

↓
↓
↓
↓
↓
↓

$\text{evalv}(M) = A_0$
$\wedge\ \text{evalv}(M) = A_1$
$\Longrightarrow A_0 = A_1$

5-5

Substitution Lemma:

If $M \hookrightarrow_v M'$ and $N \hookrightarrow_v N'$, then

$M[X \leftarrow N] \hookrightarrow_v M'[X \leftarrow N'] \rightarrow L$

$(\lambda X. M)\ N \longrightarrow (\lambda X. M')\ N'$

different!

$(\lambda X.\ M\ X) \quad \eta \quad M$

```
int f(int x) {
  return g(x);
}
```

$\Rightarrow$ g

$(\lambda X.\ 5\ X) \Rightarrow 5$

Observational Equivalence

- Do two programs do the same thing?
  - $\twoheadrightarrow_v$ does this?] — NO.
  - $\hookrightarrow_v$ does this?

(add 5 10) dist

~~(add 5 (add 5 5))~~

(+ 7 8)

$=_v$ is $\twoheadrightarrow_v$ and $\twoheadrightarrow_v^R$

for (i = 0 .. 50)
  x = i * 10
  a[i] = x

$\Rightarrow$ x = 0, for (i = 0 .. 50)
  x = x + 10
  a[i] = x

[$(\lambda X.\ M)\quad A)$

cmp

$((\lambda Y.\ N)\quad A)$

5-4

A context ::= a program with a hole

$$(\blacksquare \quad A) = C$$

fill the hole in C with X, $C[X]$

$$(X \quad A)$$

$C = \blacksquare$	$\blacksquare[N] = N$
$\mid (\lambda X.\ C)$	$(\lambda X. C)[N] = (\lambda X.\ C[N])$
$\mid (C \quad M)$	$(C\ M)[N] = (C[N] \quad M)$
$\mid (M \quad C)$	$(M\ C)[N] = (M \quad C[N])$
$\mid (o^n \quad M \ldots \quad C\ M \ldots)$	$[N] = (o^n \quad M \ldots \ C[N]\ M \ldots)$

"observational equivalence" of M and N is ~~tb~~ $\simeq$

$$\forall C.\quad C[M] =_v C[N]$$

$(\blacksquare \quad 5)$ $\quad (\blacksquare \quad +)$ $\qquad (\lambda X. \Omega) \simeq (\lambda Y. (\Omega\ \Omega))$

$(\blacksquare \quad 8)$ $\quad \blacksquare$

(put it in a list, reverse list, get last elem, then call it)

$\rightarrow_v$ compatible closure

$$C[(\lambda X. M)\ V] \rightarrow_v C[M[X \leftarrow V]]$$
$$C[(o^n\ V_1 \ldots V_n)] \rightarrow_v C[\delta(o^n, V_1, \ldots, V_n)]$$

$(+ \ (+\ 7\ 8) \ (+\ 1\ 2))$

$=$

$(+ \ (+\ 7\ 8) \quad \blacksquare)[(+\ 1\ 2)] \rightarrow_v (+\ (+\ 7\ 8)\ \blacksquare)[3]$

$=$

$(+\ (+\ 7\ 8)\ 3)$