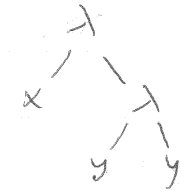
5-1

# The lambda calculus — by Alonzo Church

$$
\begin{aligned}
M, N, L \; = \;& X && \text{— a variable} \\
\mid\;& (\lambda X.\, M) && \text{— an abstraction } \lambda \\
\mid\;& (M\ N) && \text{— an application — @} \\
X \; = \;& \text{a "variable"} && \text{— } x, y, z, a, b, c, \ldots
\end{aligned}
$$

$$\underbrace{((((\lambda x.x)\ (\lambda y.\lambda x.y))\ (\lambda y.y))}_{M}\ \underbrace{(\lambda z.z))}_{N}$$


$$((\lambda x.\, M)\ N) \to_\beta \underbrace{M[X \leftarrow N]}_{\text{find the Xs in M and turn them into N}}$$

~~λx.λy.x~~

$$f(x) = 5\cdot x + x$$

$$f(6) = 5\cdot 6 + 6 = 36$$

$$\underbrace{M = 5\cdot x + x \quad [x \leftarrow 6]}_{\text{substitution}}$$

$$f(3+3)$$

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$$
\begin{aligned}
X_1\,[X_1 \leftarrow M] &= M \\
X_2\,[X_1 \leftarrow M] &= X_2 \qquad (\text{assm } X_1 \neq X_2) \\
(N\ L)\,[X_1 \leftarrow M] &= (N[X_1 \leftarrow M]\ \ L[X_1 \leftarrow M]) \\
(\lambda X_1.\, N)\,[X_1 \leftarrow M] &= \lambda X_1.\, N \qquad \text{what if } X_2 \in M\text{?} \\
(\lambda X_2.\, N)\,[X_1 \leftarrow M] &= \cancel{\lambda X_2.\ N[X_1 \leftarrow M]} \\
&\phantom{=}\ \lambda X_3.\, N[X_2 \leftarrow X_3][X_1 \leftarrow M]
\end{aligned}
$$

$X_3$ not mentioned in N or M

3-2

Beta: $((\lambda X. N)\ M)\ \ \beta\ \ \cancel{\ }\ N[X \leftarrow M]$

Alpha: $(\lambda X. M)\ \ \alpha\ \ (\lambda Y. M[X \leftarrow Y])$

for $Y \notin M$

Eta: $\cancel{(\lambda X. (M}$

$(\lambda X. (M\ X))\ \ \eta\ \ M$

$n = \beta \cup \alpha \cup \eta$

$\rightarrow_n$

$\twoheadrightarrow_n$

$=_n$

$eval_n$

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Booleans — exist to be "if"d

true := $\lambda x. \lambda y. x$

false := $\lambda x. \lambda y. y$

if M N L := $((M\ N)\ L)$

if true A B $\cancel{\ }\ \twoheadrightarrow_n$ A

if false A B $\twoheadrightarrow_n$ B

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Pairs — pair, fst, snd

fst (pair A B) $\twoheadrightarrow_n$ A

snd (pair A B) $\twoheadrightarrow_n$ B

fst M := (M true)

snd M := (M false)

pair A B := $\lambda s.$ if s A B

3-31

Numbers — what do they do?
— they iterate

$\lambda f.\ f$ $\qquad$ zero := $\lambda f. \lambda z. z$.

one := $(\lambda f. \lambda z.\ f\ z)$

two := $(\lambda f. \lambda z.\ f\ (f\ z))$

$$\text{add1} = \lambda n.\ \lambda f. \lambda z.\ f\ ((n\ f)\ z)$$

$$\text{plus} = \lambda n. \lambda m.\ \lambda f. \lambda z.\ (n\ f)\ ((m\ f)\ z)$$

$$(m\ f)\ ((n\ f)\ z)$$

$$\text{iszero} = \lambda n.\ n\ \underbrace{(\lambda x. \text{false})}\ \underbrace{\text{true}}$$

Church Numerals

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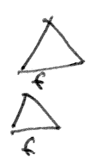
$$f := \lambda x.\ \text{if}\ \ \text{iszero}\ x$$

$$\text{zero}$$

$$\text{add}\ \ x\ \ (f\ \ (\text{sub1}\ x))$$




$$\text{mkf} := \lambda f.\ \lambda x.\ \ldots$$

$$\text{mkf}' := \lambda \text{mkf}'.\ \lambda x.\ \ldots\ \ ((\text{mkf}'\ \text{mkf}')\ (\text{sub1}\ x))$$

$$f := (\text{mkf}'\ \text{mkf}')$$

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$$\text{mkmk} := \lambda k. \lambda t.\ t\ ((k\ k)\ t)$$

$$\text{mk} := (\text{mkmk}\ \ \text{mkmk})$$

$$m\ (\text{mk}\ m) = (\text{mk}\ \ m) \qquad \forall m.$$

$$= ((\text{mkmk}\ \ \text{mkmk})\ \ m)$$

$$= (\lambda t.\ t\ ((\text{mkmk}\ \text{mkmk})\ t))\ \ m$$

$$= m\ ((\text{mkmk}\ \ \text{mkmk})\ \ m)$$

3-4/ Infinite Loop

$$M \to M' \to M'' \to \dots$$

(divergence)

$$M \to M' \to \dots \to M$$

(loop)

$$\Omega = ((\lambda x.\ (x\ x))\ \ (\lambda x.\ (x\ x)))$$

$$(\ (\lambda x.\ (x\ x))\ \ \underbrace{(\lambda y.\ (y\ y))}\ )$$

$$((\lambda y.\ (y y))\ \ (\lambda y.\ (y\ y)))$$

$$= \Omega$$