91.304 Foundations of (Theoretical) Computer Science

Chapter 4 Lecture Notes (Section 4.2: The "Halting" Problem)

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With modifications by Prof. Karen Daniels, Fall 2014

Back to Σ_1

- So the fact that Σ_1 is not closed under complement means that there exists some language L that is not recognizable by any TM
- By Church-Turing thesis this means that *no imaginable finite computer,* even with infinite memory, could recognize this language L!

Non-recognizable languages

- We proceed to prove that non-Turing recognizable languages exist, in two ways:
  - A **nonconstructive** proof using Georg Cantor's famous 1873 diagonalization technique, and then
  - An **explicit construction** of such a language.

Learning how to count

- **Definition** Let A and B be sets. Then we write A ≈ B and say that A is **equinumerous** to B if there exists a one-to-one, onto function (a "correspondence", i.e. a pairing)

  $f : A \to B$

- Note that this is a purely mathematical definition: the function *f* does not have to be expressible by a Turing machine or anything like that.
- **Example:** { 1, 3, 2 } ≈ { six, seven, BBCCD }
- **Example:** **N** ≈ **Q** (textbook example 4.15)
  - See next slide...

Learning how to count (continued)

- □ **Example: N ≈ Q** (textbook example 4.15)

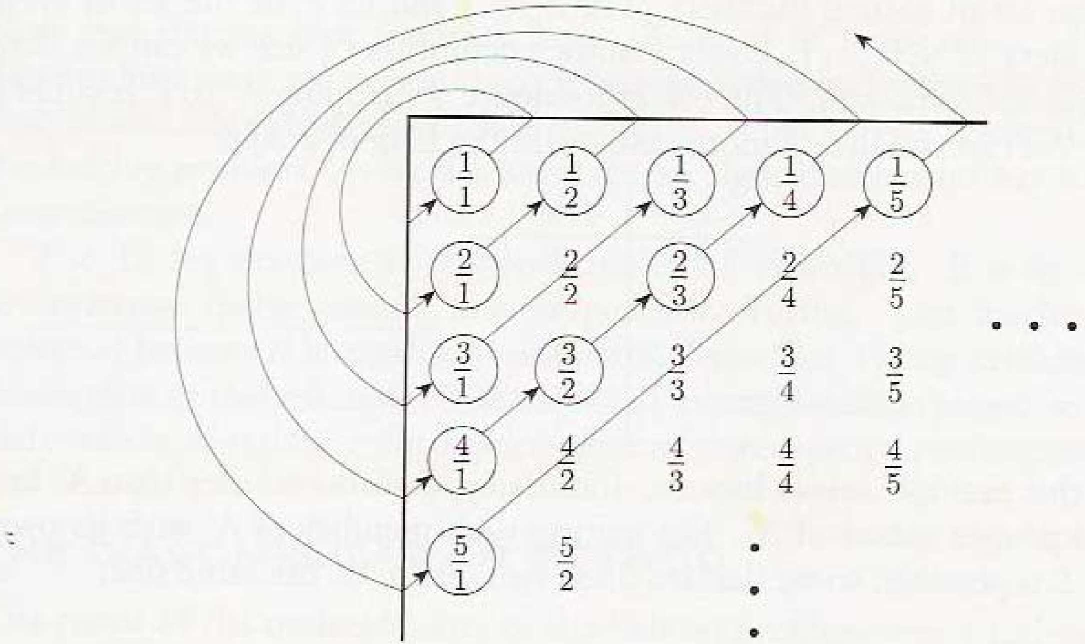


FIGURE 4.16
A correspondence of $\mathcal{N}$ and $\mathcal{Q}$

Source: Sipser textbook

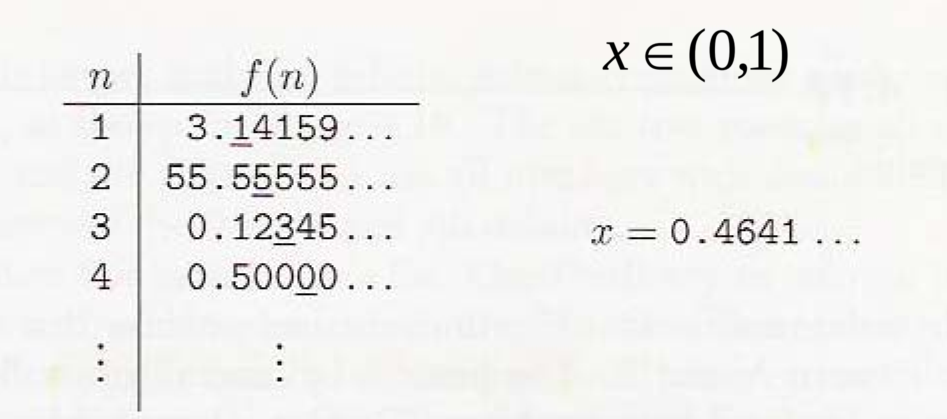
Countability

- **Definition** A set S is **countable** if S is finite **or** S ≈ **N**.
  - Saying that S is countable means that you can line up all of its elements, one after another, and cover them all
  - Note that **R** is *not* countable (Theorem 4.17), basically because choosing a single real number requires making infinitely many choices of what each digit in it is (see next slide).

Countability (continued)

- □ Theorem 4.17: **R** is *not* countable.
- □ Proof Sketch: By way of contradiction, suppose **R** ≈ **N** using correspondence *f*.

  Construct $x \in$ **R** such that x is not paired with anything in **N**, providing a contradiction.



$x \in (0,1)$

n	$f(n)$
1	$3.\underline{1}4159\ldots$
2	$55.5\underline{5}555\ldots$
3	$0.12\underline{3}45\ldots$
4	$0.500\underline{0}0\ldots$
$\vdots$	$\vdots$

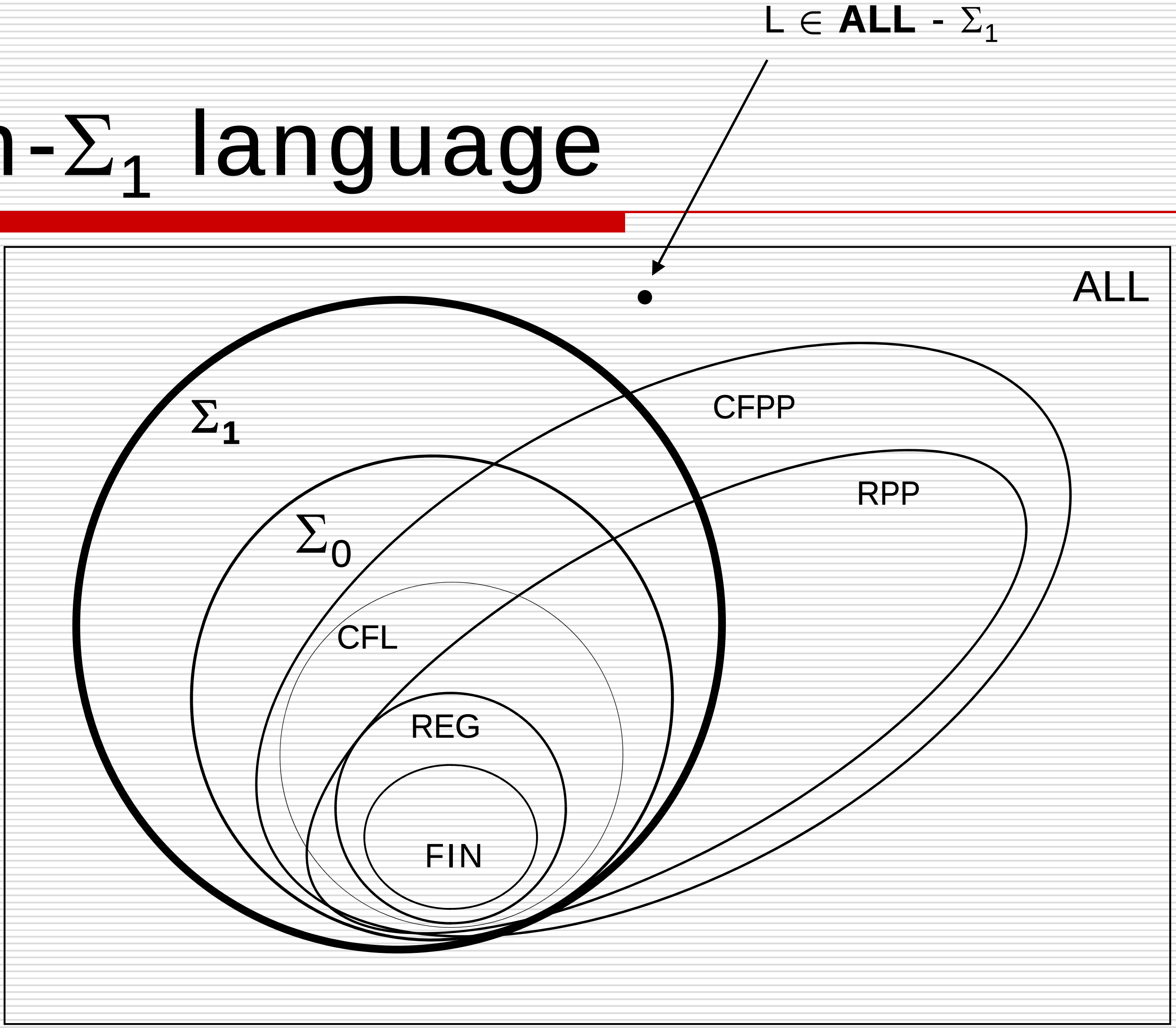
$x = 0.4641\ldots$

x is not $f(n)$ for any n because it differs from $f(n)$ in nth fractional digit.

Caveat: How to circumvent 0.1999…= 0.2000… problem?

Source: Sipser textbook

A non-Σ_1 language



Each point is a language in this Venn diagram

Strategy

- We'll show that there are more (a *lot* more) languages in ALL than there are in Σ_1
  - Namely, that Σ_1 is countable but ALL isn't countable
  - Which implies that $\Sigma_1 \neq$ ALL
  - Which implies that there exists some L that is not in Σ_1
- For simplicity and concreteness, we'll work in the universe of strings over the alphabet {0,1}.

Countability of Σ_1

- **Theorem** Σ_1 is countable
- **Proof** The strategy is simple. Σ_1 is the class of all languages that are Turing-recognizable. So each one has (at least) one TM that recognizes it. We'll concentrate on listing those TMs.

Countability of TM

- Let **TM** = { <M> | M is a Turing Machine with $\Sigma=\{0,1\}$ }
  - Notation: <M> means the **string encoding** of the object M
  - Previously, we thought of our TMs as abstract mathematical things: drawings on the board, or 7-tuples: $(Q,\Sigma,\Gamma,\delta,q_0,q_a,q_r)$
  - But just as we can encode every C++ program as an ASCII string, surely we can also encode every TM as a string
  - It's not hard to specify precisely how to do it—but it doesn't help us much either, so we won't bother
  - Just note that in our full specification of a TM $(Q,\Sigma,\Gamma,\delta,q_0,q_a,q_r)$, each element in the list is finite by definition
  - So writing down the sequence of 7 things can be done in a finite amount of text
    - In other words, each <M> is a string

Countability of TM

- Now we make a list of all possible strings in lexicographical (string) order,
- Cross out the ones that are not valid encodings of Turing Machines,
- And we have a mapping $f : \mathbf{N} \to \mathbf{TM}$
  - $f(1)$ = first (smallest) TM encoding on list
  - $f(2)$ = second TM encoding on list
  - …
- This is part of textbook's proof of Corollary 4.18 (*Some languages are not Turing-recognizable*).

Back to countability of Σ_1

- Now consider the list $L(f(1)), L(f(2)), \cdots$
  - Turns each TM enumerated by f into a language
  - So we can define a function $g : \mathbf{N} \rightarrow \Sigma_1$ by $g(i) = L(f(i))$, where $f(i)$ returns the i^{th} Turing machine
  - Now: is this a correspondence? Namely,
    - Is it onto?
    - Is it one-to-one?

Fixing g : $\mathbf{N} \rightarrow \Sigma_1$

- Go ahead and make the list $g(1), g(2), \cdots$
- But cross out each element that is a repeat, removing it from the list
  - Subtlety regarding EQ_{TM} undecidability (Ch 5)
- Then let $h : \mathbf{N} \rightarrow \Sigma_1$ be defined by

  h(i) = the i^{th} element on the reduced list
- Then h is both one-to-one and onto
- **Thus Σ_1 is countable**

What about ALL?

- **Theorem** (Cantor, 1873) For *every* set A, $A \not\approx \mathcal{P}(A)$
  - See next several slides for proof.
  - See textbook for a different way to show ALL is uncountable using *characteristic sequence* associated with (uncountable) set of all infinite binary sequences.
- Remember ALL = $\mathcal{P}(\{0,1\}^*)$ if alphabet $\Sigma = \{0,1\}$
  - set of all (languages)
    = set of all (subsets of $\{0,1\}^*$)
- Note that $\{0,1\}^*$ *is* countable
  - Just list all of the strings in lexicographical order
- **Corollary to Theorem** ALL = $\mathcal{P}(\{0,1\}^*)$ is uncountable
  - So Σ_1 is countable but ALL isn't
  - So they're not equal

Cantor’s Theorem

Theorem For every set A, $A \not\approx \mathcal{P}(A)$

Proof We’ll show by contradiction that no function $f:A\to\mathcal{P}(A)$ is onto. So suppose $f:A\to\mathcal{P}(A)$ is onto. We define a set $K\subseteq A$ in terms of it:

$$K=\{\ x \in A \mid x \notin f(x)\ \}$$

Since $K\subseteq A$, $K\in\mathcal{P}(A)$ as well (by definition of $\mathcal{P}$). Since f is onto, there exists some $z\in A$ such that $f(z) = K$. Looking closer,

Case 1: If $z\in K \underset{\text{by definition of K}}{\Rightarrow} z \notin f(z) \underset{\text{by definition of z}}{\Rightarrow} z\notin K$

so $z\in K$ certainly can’t be true…

Cantor’s Theorem

$$\text{unchanged} \quad \left\{ \begin{array}{l} K = \{\ x \in A \mid x \notin f(x)\ \} \\ K \in \mathcal{P}(A) \\ z \in A \text{ and } f(z) = K \end{array} \right.$$

On the other hand,

Case 2: If $z \notin K \underset{\substack{\uparrow \\ \text{by definition of K}}}{\Rightarrow} z \in f(z) \underset{\substack{\nearrow \\ \text{by definition of z}}}{\Rightarrow} z \in K$

so $z \notin K$ can’t be true either! **QED**

Cantor’s Theorem: Example

- □ For every *proposed* f : A→$\mathcal{P}$(A), the theorem constructs a set K∈$\mathcal{P}$(A) that is not f(x) for any x
- □ Let A = { 1, 2, 3 }
  $\mathcal{P}$(A) = { ∅, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3} }
- □ Propose f : A→$\mathcal{P}$(A), show K

Diagonalization

- □ All we're really doing is identifying the squares on the diagonal and making them different than what's in our set K
- □ So that we're guaranteed K ≠ f(1), K ≠ f(2), …
- □ The construction works for infinite sets too

x	f(x)
1	{ 🟥, _ , _ }
2	{ _ , 🟥, _ }
3	{ _ , _ , 🟥 }

Non-recognizable languages

- □ So we conclude that there exists some L ∈ ALL - Σ_1 (**many** such languages)
- □ But we don’t know what any L looks like exactly
- □ Turing constructed such an L also using diagonalization (but not the $\not\approx$ relation)
- □ We now turn our attention to it

Programs that process programs

- In §4.1, we considered languages such as A_{CFG} = { <G,w> | G is a CFG and w∈L(G) }
- Each element of A_{CFG} is a *coded pair*
  - Meaning that the grammar G is encoded as a string **and**
  - w is an arbitrary string **and**
  - <G,w> contains both pieces, in order, in such a way that the two pieces can be easily extracted
- The question "does grammar G_1 generate the string 00010?" can then be phrased equivalently as:
  - Is $<G_1,00010> \in A_{CFG}$?

Programs that process programs

- Prelude to introducing Universal TM that can "process" programs.
- A_{CFG} = { <G,w> | G is a CFG and w∈L(G) }
- The *language* A_{CFG} somehow represents the question "does *this* grammar accept *that* string?"
- **Additionally** we can ask: is A_{CFG} itself a regular language? context free? decidable? recognizable?
  - We showed previously that A_{CFG} is decidable (as is almost everything similar in §4.1)

A_{TM} and the Universal TM

- A_{TM} = { <M,w> | M is a TM and w∈L(M) }
- We will show that $A_{TM} \in \Sigma_1 - \Sigma_0$
  - (It's recognizable but not decidable)
- **Theorem** A_{TM} is Turing-recognized by a fixed TM called U (the Universal TM)
  - This is not stated as a theorem in the textbook (it does appear as part of proof of *Theorem 4.11: A_{TM} is undecidable*), but should be: it's really important

A_{TM} = L (U)

A_{TM} = { <M,w> | M is a TM and $w \in L(M)$ }

U is a 3-tape TM that keeps data like this:

1	< M >	*never changes*
2	q	*a state name*
3	$c_1\ c_2\ c_3 \cdots$	*tape contents & head pos*

On startup, U receives input <M,w> and writes <M> onto tape 1 and w onto tape 3. (If the input is not of the form <M,w>, then U rejects it.) From <M>, U can extract the encoded pieces $(Q,\Sigma,\Gamma,\delta,q_0,q_{acc},q_{rej})$ at will. It continues by extracting and writing q_0 onto tape 2.

A_{TM} = L (U)

$A_{TM} = \{ \langle M,w \rangle \mid M \text{ is a TM and } w \in L(M) \}$

1	< M >	*never changes*
2	q	*a state name*
3	$c_1\ c_2\ c_3 \cdots$	*tape contents & head pos*

To simulate a single computation step, U fetches the current character **c** from tape 3, the current state **q** on tape 2, and looks up the value of $\delta(q,c)$ on tape 1, obtaining a new state name, a new character to write, and a direction to move. U writes these on tapes 2 and 3 respectively.

If the new state is q_{acc} or q_{rej} then U accepts or rejects, respectively. Otherwise it continues with the next computation step.

The Universal TM U

- This U is **hugely important**: it's the theoretical basis for *programmable* computers.
- It says that there is a *fixed* machine U that can take computer programs as *input* and behave just like each of those programs
  - Note that U is **not** a decider
  - See VMware
- Since A_{TM} = L(U), we have shown that A_{TM} is Turing-recognizable (Σ_1)

The “Halting” Problem

- A_{TM} = {<M,w> | M is a TM and w $\in$L(M)}
- This appears in our textbook as:
  - A_{TM} = { <M,w> | M is a TM and M accepts w }
  - This emphasizes the fact that U might loop (i.e. might not halt) on input <M,w>.
  - A_{TM} is therefore sometimes called the halting problem.
  - We use “” here due to Chapter 5’s discussion…
    - A_{TM} is called the acceptance problem in Chapter 5
    - The “real” halting problem is defined there as:
      - $HALT_{TM}$ = { <M,w> | M is a TM and M halts on input w }

A_{TM} is undecidable

Theorem 4.11 (Turing) $A_{TM} \notin \Sigma_0$

Proof Suppose that $A_{TM}=L(H)$ where H is a decider. We'll show that this leads to a contradiction.

$$H(\langle M,w\rangle) = \begin{cases} \textit{accept} & \text{if M accepts w} \\ \text{reject} & \text{if M does not accept w} \end{cases}$$

Let **D** be a TM that behaves as follows:

1. Input x
2. If x is not of the form <M> for some TM M, then D rejects
3. Simulate H on input <M, <M> >
   - □ If H accepts <M, <M>>, then D rejects
   - □ If H rejects <M, <M>>, then D accepts

Input M's description!

Do the opposite!

“Simulate H”

- Steps 1 and 2 are not so hard to imagine
- How does D “simulate H on (some other input)”?
  - If someone creates an H, we follow this outline to build D — which has the entire H program built in as a subroutine
  - Note we run H on a *different* input than the one that D is given
- Also, we didn’t say what D does if H goes into an infinite loop
  - It’s OK because H does *not* do that, by the assumption that H is a decider

Language accepted by D

(Repeat) **D** behaves as follows:

1. D: input x
2. if x is not of the form <M> for some TM M, then D rejects
3. simulate H on input <M, <M> >
   - □ If H accepts <M, <M>>, then D rejects
   - □ If H rejects <M, <M>>, then D accepts

So L(D)={ <M> | H rejects <M,<M>> }

Now H is a recognizer (even a decider) for A_{TM}, so if H rejects <M,<M>> then it means that the machine M **does not accept** <M>.

So L(D)={ <M> | <M> ∉ L(M) }

Impossible machine

- So $L(D) = \{ <M> \mid <M> \notin L(M) \}$
- What if we give a copy of D’s own description <D> to itself as input? As in Cantor’s theorem, we have trouble:
  - $<D> \in L(D) \Rightarrow <D> \notin L(D)$!!
  - $<D> \notin L(D) \Rightarrow <D> \in L(D)$!!
- So this D can’t exist. But it was defined as a fairly straightforward wrapper around H: so H must not exist either. That is, there is no decider for A_{TM}. **QED**

To summarize...

H accepts <M,w> exactly when M accepts w.

⇩

D rejects <M> exactly when M accepts <M>.

⇩

D rejects <D> exactly when D accepts <D>.

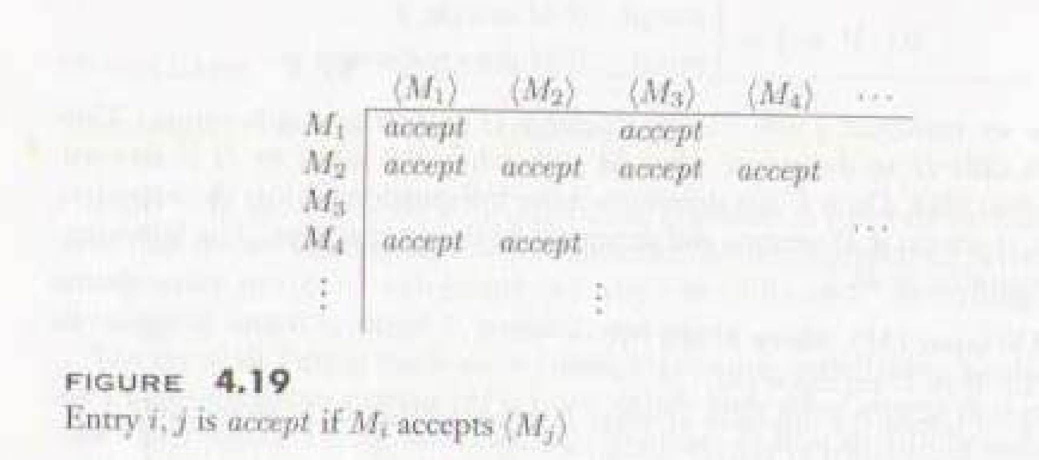
contradiction!

Diagonalization in this proof?

M_i is a TM.

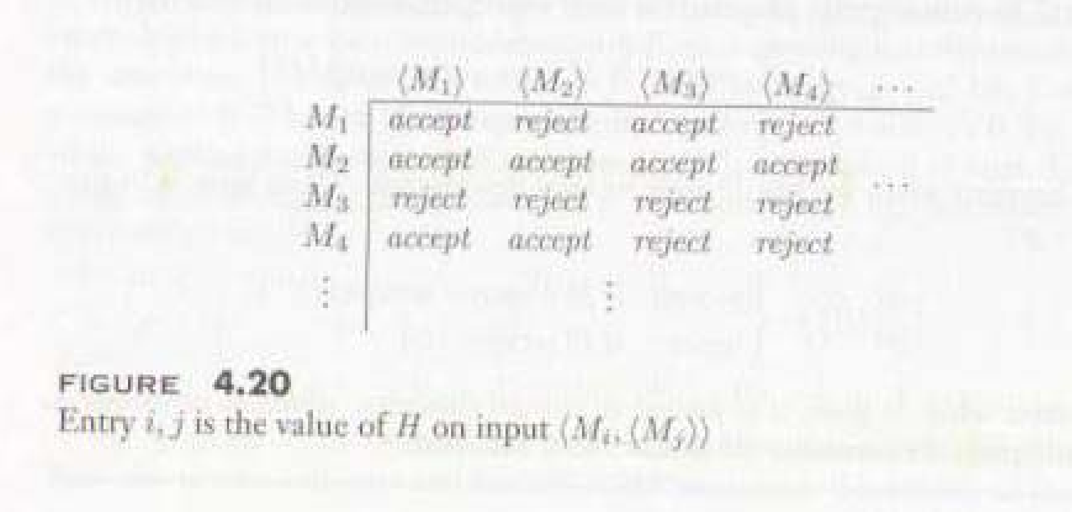
Blank entry implies either loop or reject.

Now consider H, which is a decider.



	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*		*accept*		
M_2	*accept*	*accept*	*accept*	*accept*	
M_3					$\cdots$
M_4	*accept*	*accept*			
$\vdots$		$\vdots$			

FIGURE 4.19
Entry i, j is *accept* if M_i accepts $\langle M_j \rangle$



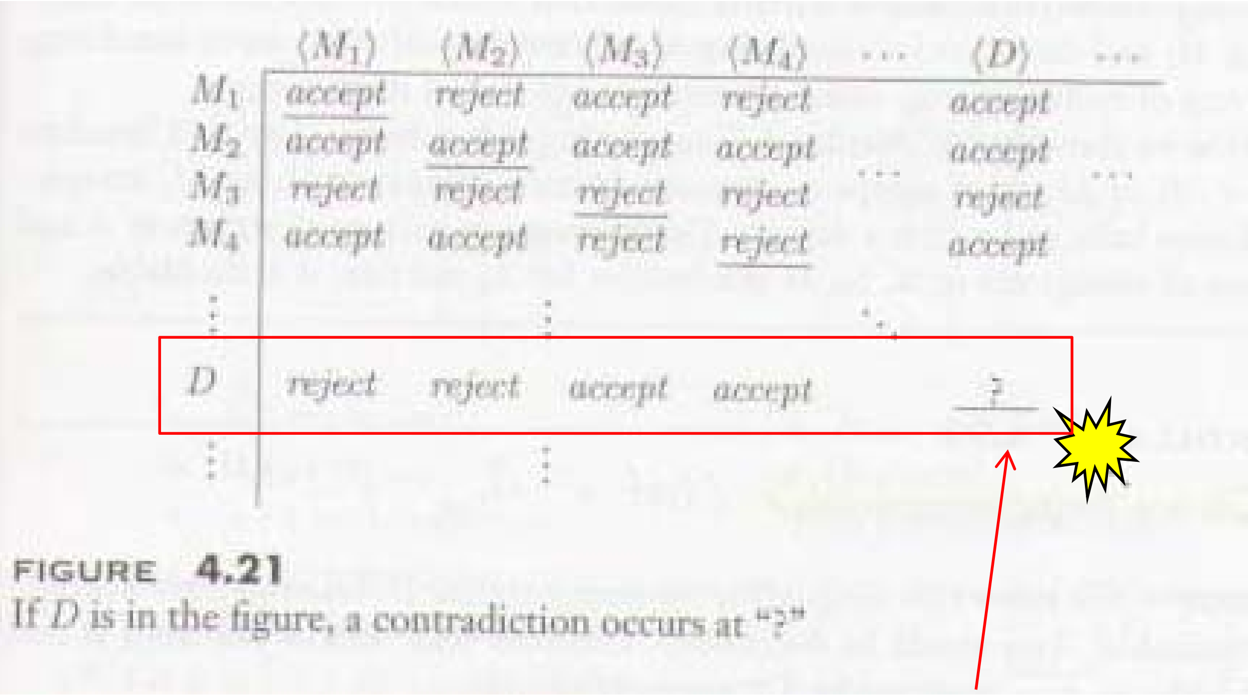
	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	
M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$		$\vdots$			

FIGURE 4.20
Entry i, j is the value of H on input $\langle M_i, \langle M_j \rangle \rangle$

Source: Sipser textbook

Diagonalization in this proof? (cont.)

D computes the opposite of each diagonal entry because its behavior is opposite H’s behavior on input $<M_i,<M_i>>$.



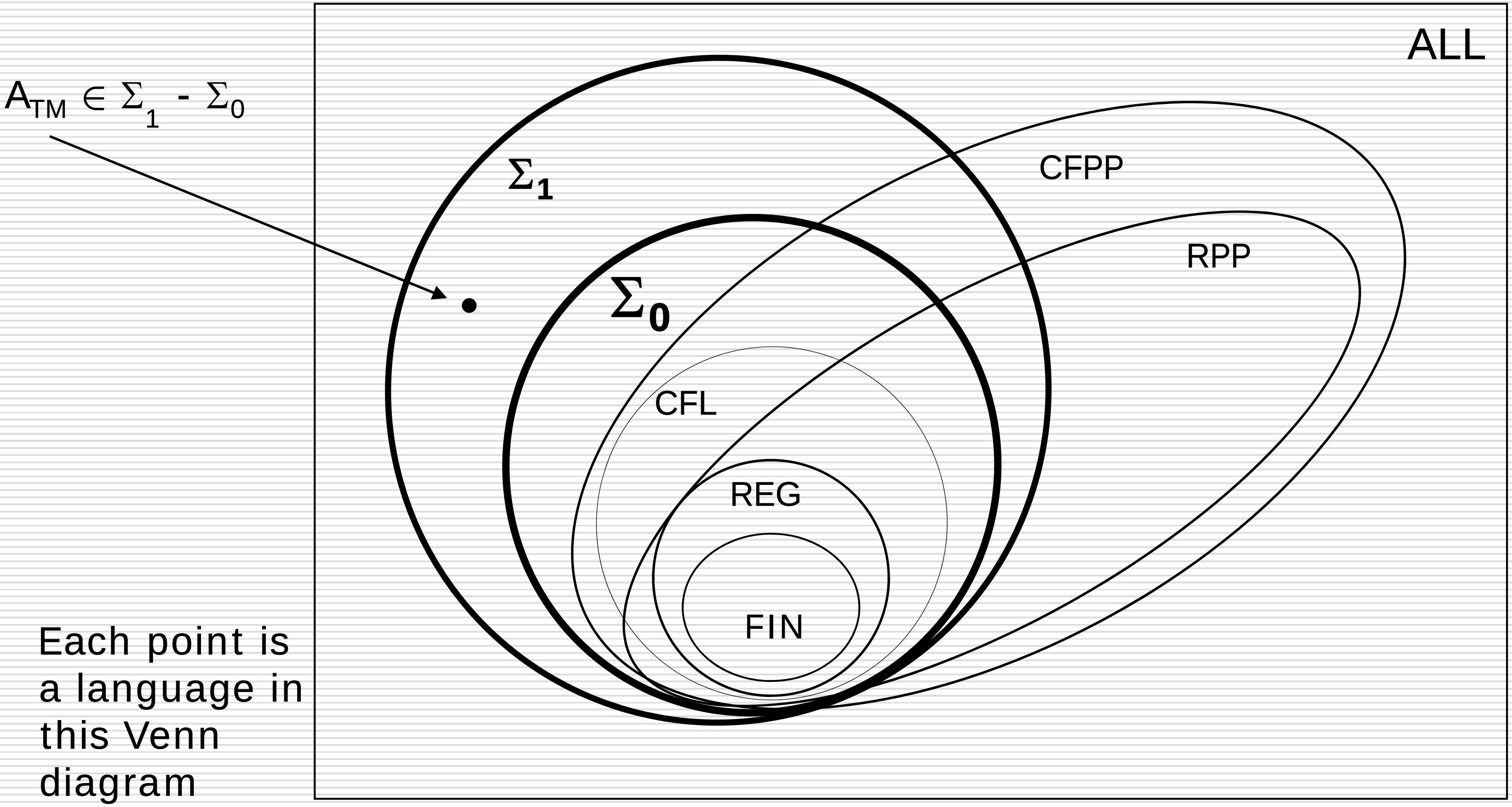
	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*		*accept*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*		*reject*	
M_4	*accept*	*accept*	*reject*	*reject*		*accept*	
$\vdots$		$\vdots$			$\ddots$		
D	*reject*	*reject*	*accept*	*accept*		?	
$\vdots$		$\vdots$					

FIGURE 4.21
If D is in the figure, a contradiction occurs at “?”

Cannot compute opposite of this entry itself!

Source: Sipser textbook

Current landscape



Each point is a language in this Venn diagram

Decidability versus recognizability

Theorem 4.22 For every language L, $L \in \Sigma_0 \Leftrightarrow (L \in \Sigma_1$ **and** $L^c \in \Sigma_1)$

Recall that complement of a language is the language consisting of all strings that are not in that language.

Proof The $\Rightarrow$ direction is easy, because $\Sigma_0 \subseteq \Sigma_1$ and Σ_0 is closed under complement.

For the $\Leftarrow$ direction, suppose that $L \in \Sigma_1$ and $L^c \in \Sigma_1$. Then there exist TMs so that $L(M_{\mathbf{1}})=L$ and $L(M_{\mathbf{2}})=L^{\mathbf{c}}$. To show that $L \in \Sigma_{\mathbf{0}}$, we need to produce a *decider* $M_{\mathbf{3}}$ such that $L=L(M_{\mathbf{3}})$.

Theorem 4.22 continued

$L(M_1)=L$, $L(M_2)=L^c$, and we want a *decider* M_3 such that $L=L(M_3)$
Strategy: given an input x, we know that either $x \in L$ or $x \in L^c$. So M_3 does this:

1. M_3: input x
2. set up tape #1 to simulate M_1 on input x and tape #2 to simulate M_2 on input x
3. compute one transition step of M_1 on tape 1 and one transition step of M_2 on tape 2
   - □ if M_1 accepts, then M_3 accepts
   - □ if M_2 accepts, then M_3 rejects
   - □ else goto 3

This is like running both M_1 and M_2 *in parallel*.

Theorem 4.22 conclusion

- For each string x, either M_1 accepts x or M_2 accepts x, but never both
  - So the machine M_3 will always halt eventually in step 3
  - Therefore, M_3 is a decider
- M_3 accepts those strings in L and rejects those strings in L^c
  - So $L(M_3) = L$ **QED**

Getting a non-recognizable language from A_{TM}

- $L \in \Sigma_0 \Leftrightarrow (L \in \Sigma_1$ **and** $L^c \in \Sigma_1)$
- $L \notin \Sigma_0 \Leftrightarrow (L \notin \Sigma_1$ **or** $L^c \notin \Sigma_1)$
- Now since we know that $A_{TM} \notin \Sigma_0$, and we know that $A_{TM} \in \Sigma_1$, it must be true that $A_{TM}{}^c \notin \Sigma_1$.
  - A_{TM} = { <M,w> | M is a TM and $w \in L(M)$ }
  - $A_{TM}{}^c$ = { x | x is not of the form <M,w> **or** (x=<M,w> and $w \notin L(M)$) }
- If we narrow this down to strings of the form <M,w>, then the language is still unrecognizable:
  - NA_{TM} = { <M,w> | M is a TM and $w \notin L(M)$ }

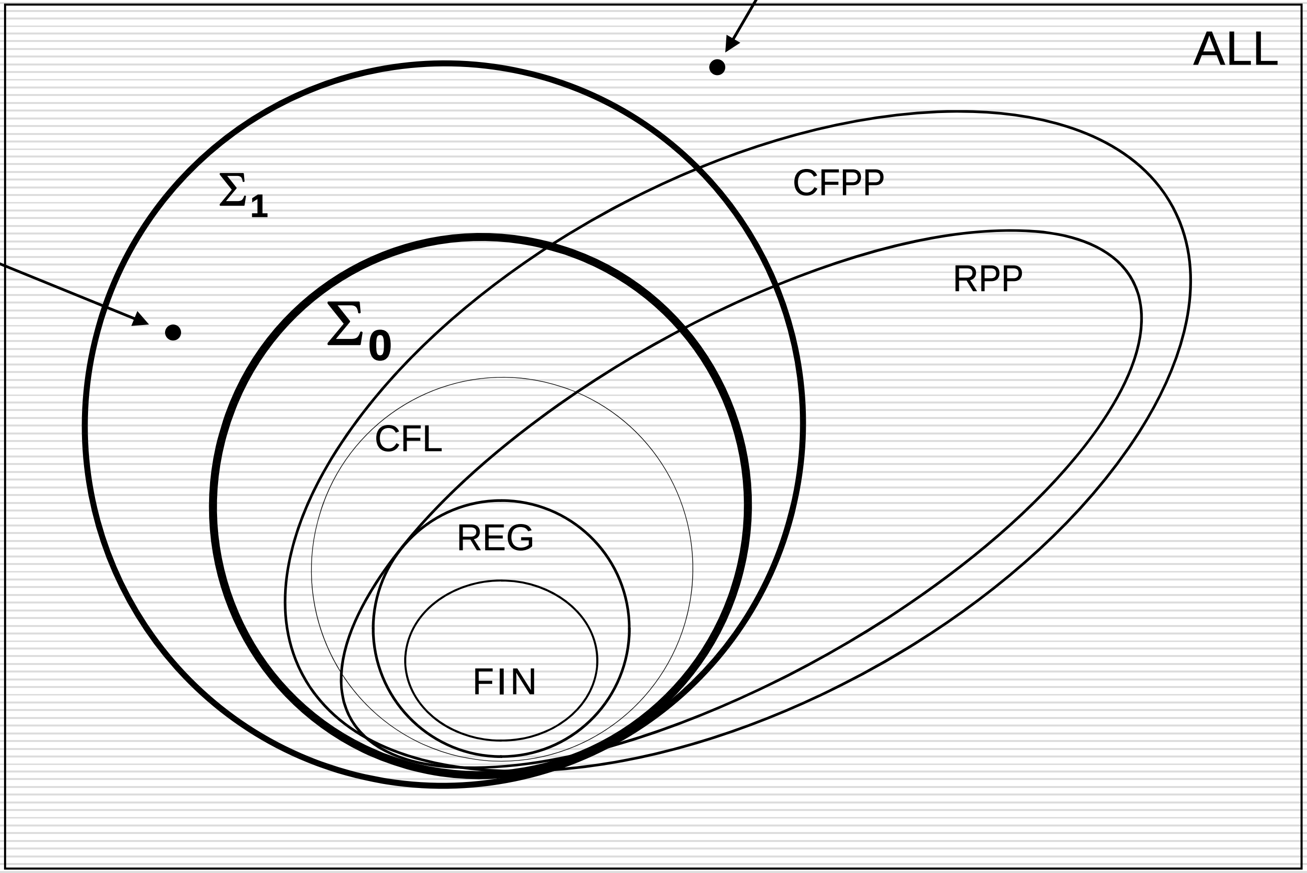
Unrecognizability

- NA_{TM} = { <M,w> | M is a TM and w $\notin$ L(M) }
- What does it mean that NA_{TM} is unrecognizable?
  - Every TM recognizes a language that's different than NA_{TM}
  - Either it accepts strings that are not in NA_{TM}, or it fails to accept some strings that actually are in NA_{TM}
- Analogy to C programs:
  - Write a C program that takes another C program as input and prints out "loop" if the other C program goes into an infinite loop.

Current landscape

$NA_{TM} \in ALL - \Sigma_1$

$A_{TM} \in \Sigma_1 - \Sigma_0$



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