91.304 Foundations of (Theoretical) Computer Science

Chapter 4 Lecture Notes (Section 4.1: Decidable Languages)

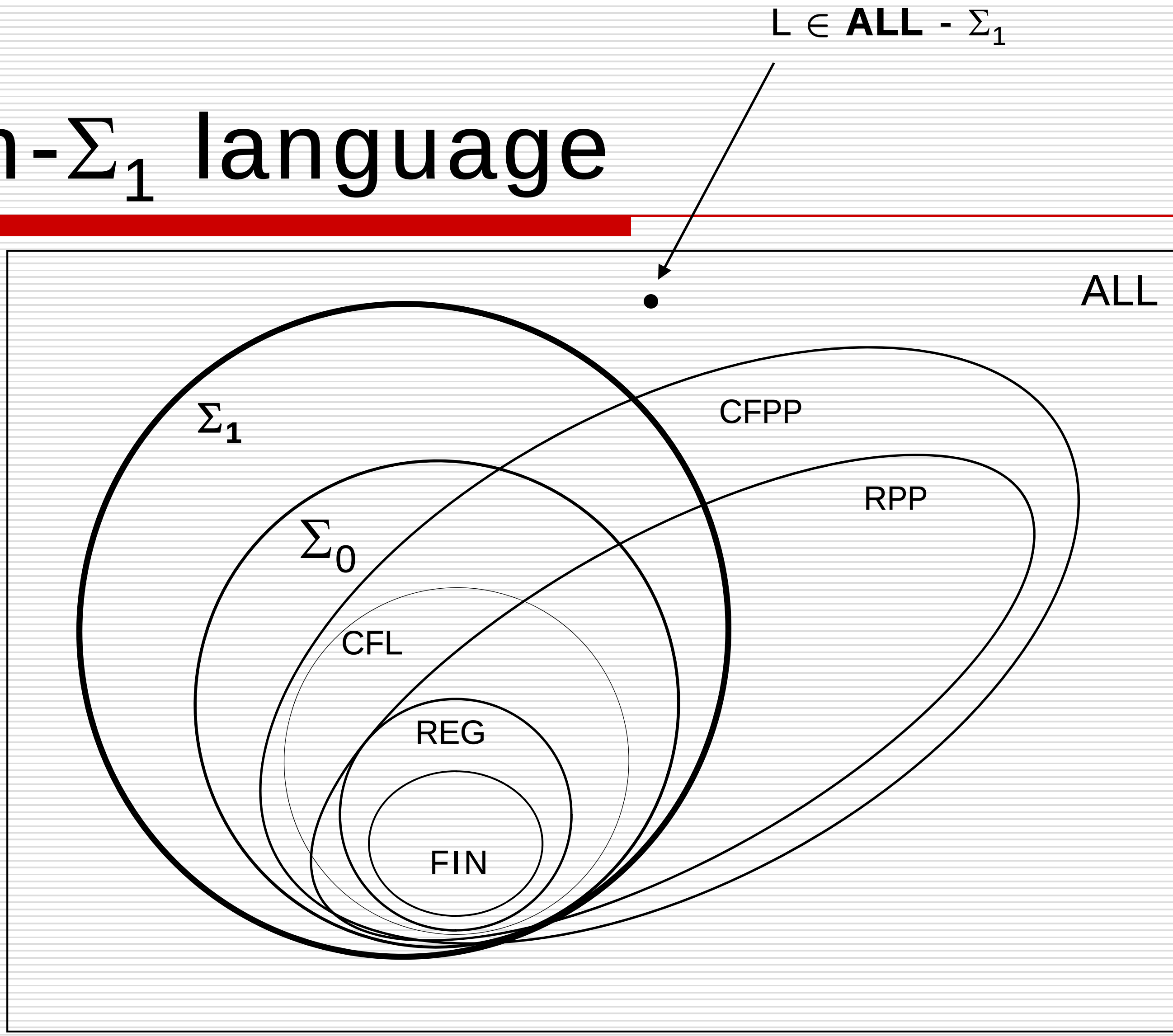
David Martin
dm@cs.uml.edu

With modifications by Prof. Karen Daniels, Fall2012

Back to Σ_1

- The fact that Σ_1 is not closed under complement means that there exists some language L that is not recognizable by any TM.
- By Church-Turing thesis this means that *no imaginable finite computer*, even with infinite memory, could recognize this language L!

A non-Σ_1 language



Each point is a language in this Venn diagram

Strategy

- Goal: Explore limits of algorithmic solvability.
- We'll show (later in Section 4.2) that there are more (a *lot* more) languages in ALL than there are in Σ_1
  - Namely, that Σ_1 is countable but ALL isn't countable
  - Which implies that $\Sigma_1 \neq$ ALL
  - Which implies that there exists some L that is not in Σ_1

Overview of Section 4.1

- *Decidable Languages* (in Σ_0): to foster later appreciation of undecidable languages
  - Regular Languages
    - A_{DFA}: Acceptance problem for DFAs
    - A_{NFA}: Acceptance problem for NFAs
    - A_{REX}: Acceptance problem for Regular Expressions
    - E_{DFA}: Emptiness testing for DFAs
    - EQ_{DFA}: 2 DFAs recognizing the same language
  - Context-Free Languages (see next slide)…

Overview of Section 4.1 (cont.)

- *Decidable Languages* (in Σ_0): to foster later appreciation of undecidable languages
  - Context-Free Languages
    - A_{CFG}: Does a given CFG generate a given string?
    - E_{CFG}: Is the language of a given CFG empty?
    - Every CFL is decidable by a Turing machine.

Overview of Section 4.1

- ***Decidable Languages*** (in Σ_0): to foster later appreciation of undecidable languages
  - Regular Languages
    - A_{DFA}: **Acceptance problem for DFAs**
    - Acceptance problem for NFAs
    - Acceptance problem for Regular Expressions
    - Emptiness testing for DFAs
    - 2 DFAs recognizing the same language

Decidable Problems for Regular Languages: DFAs

- **Acceptance problem for DFAs**

$$A_{DFA} = \{< B, w > | \ B \text{ is a DFA that accepts a given string } w\}$$

  - Language includes encodings of all DFAs and strings they accept.
  - Showing language is decidable is same as showing the computational problem is decidable.
- **Theorem 4.1**: A_{DFA} is a decidable language.
  - **Proof Idea**: Specify a TM M that decides A_{DFA}.
    - M = "On input <B,w>, where B is a DFA and w is a string (implicit legal encoding check too):
      1. Simulate B on input w.
      2. If simulation ends in accept state, *accept*. If it ends in nonaccepting state, *reject*."

Implementation details??

Overview of Section 4.1

- *Decidable Languages* (in Σ_0): to foster later appreciation of undecidable languages
  - Regular Languages
    - Acceptance problem for DFAs
    - A_{NFA}: **Acceptance problem for NFAs**
    - Acceptance problem for Regular Expressions
    - Emptiness testing for DFAs
    - 2 DFAs recognizing the same language

Decidable Problems for Regular Languages: NFAs

- **Acceptance problem for NFAs**

$$A_{\text{NFA}} = \{<B, w> \mid B \text{ is an NFA that accepts a given string } w\}$$

- **Theorem 4.2**: A_{NFA} is a decidable language.
  - **Proof Idea**: Specify a TM N that decides A_{NFA}.
    - N = "On input <B,w>, where B is an NFA and w is a string:
      1. Convert NFA B to equivalent DFA C using Theorem 1.39.
      2. Run TM M from Theorem 4.1 on input <C,w>.
      3. If M accepts, *accept*. Otherwise, *reject*."

N uses M as a "subroutine."

Alternatively, could we have modified proof of Theorem 4.1 to accommodate NFAs?

Overview of Section 4.1

- *Decidable Languages* (in Σ_0): to foster later appreciation of undecidable languages
  - Regular Languages
    - Acceptance problem for DFAs
    - Acceptance problem for NFAs
    - A_{REX}: **Acceptance problem for Regular Expressions**
    - Emptiness testing for DFAs
    - 2 DFAs recognizing the same language

Decidable Problems for Regular Languages: Regular Expressions

- **Acceptance problem for Regular Expressions**

$$A_{REX} = \{< R, w > | \, R \text{ is a regular expression that generates string } w\}$$

- **Theorem 4.3**: A_{REX} is a decidable language.
  - **Proof Idea**: Specify a TM P that decides A_{REX}.
    - P = "On input $<R,w>$, where R is a regular expression and w is a string:
      1. Convert regular expression R to equivalent NFA A using Theorem 1.54.
      2. Run TM N from Theorem 4.2 on input $<A,w>$.
      3. If N accepts, *accept*. If N rejects, *reject*."

Overview of Section 4.1

- ***Decidable Languages*** (in Σ_0): to foster later appreciation of undecidable languages
  - Regular Languages
    - Acceptance problem for DFAs
    - Acceptance problem for NFAs
    - Acceptance problem for Regular Expressions
    - E_{DFA}: **Emptiness testing for DFAs**
    - 2 DFAs recognizing the same language

Decidable Problems for Regular Languages: DFAs

- **Emptiness problem for DFAs**

  $$\mathrm{E}_{\mathrm{DFA}} = \{<A>|\ A \text{ is a DFA and } L(A) = \emptyset\}$$

- **Theorem 4.4**: $\mathrm{E}_{\mathrm{DFA}}$ is a decidable language.
  - **Proof Idea**: Specify a TM T that decides $\mathrm{E}_{\mathrm{DFA}}$.
    - T = "On input <A>, where A is a DFA:
      1. Mark start state of A.
      2. Repeat until no new states are marked:
      3. Mark any state that has a transition coming into it from any state that is already marked.
      4. If no accept state is marked, *accept*; otherwise, *reject*."

Example (board work)

Overview of Section 4.1

- *Decidable Languages* (in Σ_0): to foster later appreciation of undecidable languages
  - Regular Languages
    - Acceptance problem for DFAs
    - Acceptance problem for NFAs
    - Acceptance problem for Regular Expressions
    - Emptiness testing for DFAs
    - EQ_{DFA}: **2 DFAs recognizing the same language**

Decidable Problems for Regular Languages: DFAs

- **2 DFAs recognizing the same language**

$$EQ_{DFA} = \{ <A, B> \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B) \}$$

- **Theorem 4.5**: EQ_{DFA} is a decidable language.

symmetric difference:

$$L(C) = \left(L(A) \cap \overline{L(B)} \right) \cup \left(\overline{L(A)} \cap L(B) \right)$$

Recall regular languages are closed under complement, intersection, union.

emptiness:

$$L(C) = \emptyset \Leftrightarrow L(A) = L(B)$$

F = "On input $\langle A, B \rangle$, where A and B are DFAs:

1. Construct DFA C as described.
2. Run TM T from Theorem 4.4 on input $\langle C \rangle$.
3. If T accepts, *accept*. If T rejects, *reject*."

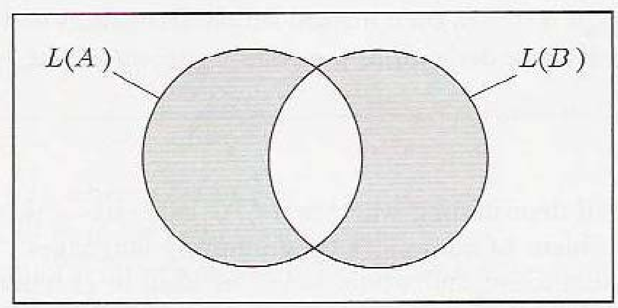


FIGURE 4.6
The symmetric difference of $L(A)$ and $L(B)$

Source: Sipser Textbook

Overview of Section 4.1

- *Decidable Languages* (in Σ_0): to foster later appreciation of undecidable languages
  - Context-Free Languages
    - A_{CFG}: **Does a given CFG generate a given string?**
    - Is the language of a given CFG empty?
    - Every CFL is decidable by a Turing machine.

Decidable Problems for Context-Free Languages: CFGs

- **Does a given CFG generate a given string?**

  $$A_{CFG} = \{<G, w> \mid G \text{ is a CFG that generates string } w\}$$

- **Theorem 4.7**: A_{CFG} is a decidable language.
  - Why is this unproductive: use G to go through all derivations to determine if any yields w?
  - Better Idea…**Proof Idea**: Specify a TM S that decides A_{CFG}.
    - S = "On input <G,w>, where G is a CFG and w is a string:
      1. Convert G to equivalent Chomsky normal form grammar.
      2. List all derivations with $2n$-1 steps (**why?**), where n = length of w. (Except if n=0, only list derivations with 1 step.)
      3. If any of these derivations yield w, *accept*; otherwise, *reject*."

Overview of Section 4.1

- *Decidable Languages* (in Σ_0): to foster later appreciation of undecidable languages
  - Context-Free Languages
    - Does a given CFG generate a given string?
    - E_{CFG}: **Is the language of a given CFG empty?**
    - Every CFL is decidable by a Turing machine.

Decidable Problems for Context-Free Languages: CFGs

- **Is the language of a given CFG empty?**

$$\mathrm{E}_{\mathrm{CFG}} = \{<G> \mid G \text{ is a CFG and } L(G) = \emptyset\}$$

- **Theorem 4.8**: $\mathrm{E}_{\mathrm{CFG}}$ is a decidable language.
  - **Proof Idea**: Specify a TM R that decides $\mathrm{E}_{\mathrm{CFG}}$.
    - R = "On input <G>, where G is a CFG:
      1. Mark all terminal symbols in G.
      2. Repeat until no new variables get marked:
      3. Mark any variable A where G has rule A->U_1 U_2 ... U_k and each symbol U_1 U_2 ... U_k has already been marked.
      1. If start variable is not marked, *accept*; otherwise, *reject*."

Decidable (?) Problems for Context-Free Languages: CFGs

- **Check if 2 CFGs generate the same language.**

$$\mathrm{EQ}_{\mathrm{CFG}} = \{<G, H> \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

- **Not decidable! (see Chapter 5)**
- Why is this possible? Why is this problem not in Σ_0 if CFL is in Σ_0?

Recall: Closure properties of CFL

- Reminder: closure properties can help us measure whether a computation model is reasonable or not
- CFL is closed under
  - Union, concatenation
    - Thus, exponentiation and *
- CFL is *not* closed under
  - Intersection
  - Complement
- Weak intersection:

  If A∈CFL and R∈**REG**, then A∩R∈ CFL

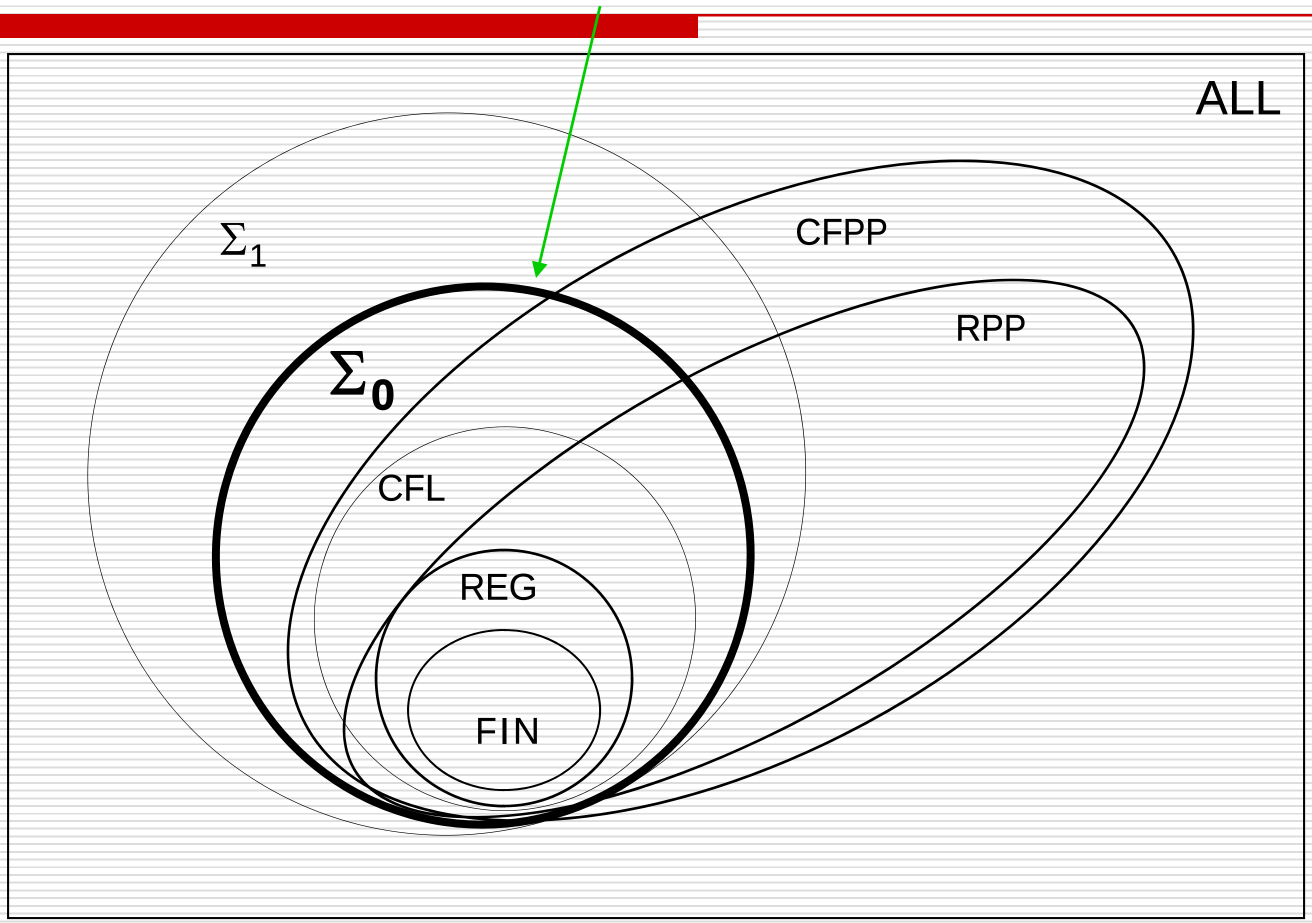
Overview of Section 4.1

- *Decidable Languages* (in Σ_0): to foster later appreciation of undecidable languages
  - Context-Free Languages
    - Does a given CFG generate a given string?
    - Is the language of a given CFG empty?
    - **Every CFL is decidable by a Turing machine.**

Decidable Problems for Context-Free Languages: CFLs

- **Every CFL is decidable by a Turing machine.**
- Bad Idea: Convert PDA for CFL into TM
- **Theorem 4.9**: Every context-free language is decidable.
  - Let A be a CFL and G be a CFG for A. (Where does G come from?)
  - Design TM M_G that decides A.
  - M_G = "On input w, where w is a string:
    1. Run TM S from Theorem 4.7 on input $<G,w>$.
    2. If S accepts, *accept*. If S rejects, *reject*."

Summary: Some problems (languages) related to languages in Σ_0 have been shown in this lecture to be in Σ_0.



Each point is a language in this Venn diagram

Remember that just because a language is in Σ_0 does **not** mean that **every** problem (language) related to members of its class is also in Σ_0!