91.304 Foundations of (Theoretical) Computer Science

Chapter 1 Lecture Notes (Section 1.3: Regular Expressions)

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with some modifications by Prof. Karen Daniels, Spring 2012

Regular expressions

- You might be familiar with these.
- Example: "^int .*\(.*\);" is a (flex format) regular expression that appears to match C function prototypes that return ints.
- In our treatment, a regular expression is a **program** that generates a **language** of matching strings when you "run it“.
- We will use a very compact definition that simplifies things later.

Flex = Fast Lexical Analyzer Generator

Regular expressions

- **Definition.** Let Σ be an alphabet not containing any of the special characters in this list: ε ∅) (∪ · ∗

  We define the syntax of the (programming) language REX(Σ), abbreviated as REX, inductively:

  - **Base cases**
    1. For all a∈Σ, a∈REX. In other words, each single character from Σ is a regular expression all by itself.
    2. ε∈REX. In other words, the literal symbol ε is a regular expression. In this context it is *not* the empty string but rather the single-character *name* for the empty string.
    3. ∅∈REX. Similarly, the literal symbol ∅ is a regular expression.

Notes:

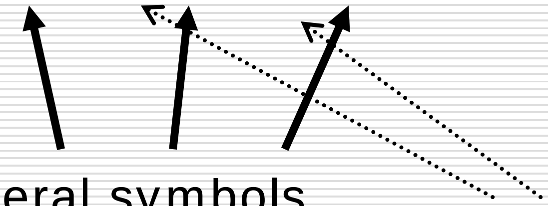
-REX is not defined in our textbook, but is helpful in continuing to build our diagram of languages.

-In our textbook, **a** represents language {a}, ε represents language {ε}.

Regular expressions

- Definition continued
  - **Induction cases**

4. For all $r_1, r_2 \in$ REX,
   $(\; r_1 \; \cup \; r_2 \;) \in$ REX also



literal symbols

variables

5. For all $r_1, r_2 \in$ REX,
   $(\; r_1 \; \cdot \; r_2 \;) \in$ REX also

Note: Later we remove dot, which is denoted by empty circle in textbook (later also removed).

Regular expressions

- Definition continued
  - Induction cases continued

    6. For all $r \in$ REX,
$(\ r^* \ ) \in$ REX also

- Examples over $\Sigma=\{0,1\}$
  - ε and 0 and 1 and $\emptyset$
  - $(((1\cdot 0)\cdot(\varepsilon\cup\emptyset))^*)$
  - $\varepsilon\varepsilon$ is *not* a regular expression
    - Remember, in the context of regular expressions, ε and $\emptyset$ are ordinary characters

Note: Textbook also defines $R^+=R\,R^*$, where R is a regular expression.

Semantics of regular expressions

- **Definition.** We define the meaning of the language REX(Σ) inductively using the L() operator so that L(r) denotes the **language generated by** r as follows:
  - **Base cases**
    1. For all $a \in \Sigma$, L(a) = { a }. *A single-character regular expression generates the corresponding single-character string.*
    2. L(ε) = { ε }. *The symbol for the empty string actually generates the empty string.*
    3. L(∅) = ∅. *The symbol for the empty language actually generates the empty language.*

Regular expressions

- Definition continued
  - **Induction cases**
    4. For all $r_1, r_2 \in$ REX,
       $L((r_1 \cup r_2)) = L(r_1) \cup L(r_2)$
    5. For all $r_1, r_2 \in$ REX,
       $L((r_1 \cdot r_2)) = L(r_1) \cdot L(r_2)$
    6. For all $r \in$ REX,
       $L((r^*)) = (L(r))^*$
  - **No other string is in REX(Σ)**
- Example
  - $L((((1\cdot 0)\cdot(\varepsilon\cup\emptyset))^*))$ includes

    ε,10,1010,101010,10101010,...

Orientation

- We used highly flexible mathematical notation and state-transition diagrams to specify DFAs and NFAs
- Now we have a precise programming language REX that generates languages
- REX is designed to **close the simplest languages under** $\cup$, $*$, $\cdot$

Abbreviations

- Instead of parentheses, we use precedence to indicate grouping when possible.
  - $*$ (highest)
  - $\cdot$
  - $\cup$(lowest)
- Instead of $\cdot$, we just write elements next to each other
  - Example: $(((1\cdot0)\cdot(\varepsilon\cup\emptyset))^*)$ can be written as $(10(\varepsilon\cup\emptyset))^*$
- If $r\in \mathrm{REX}(\Sigma)$, instead of writing rr^*, we write r^+

Abbreviations

- Instead of writing a union of all characters from Σ together to mean "any character", we just write Σ
    - In a flex/grep regular expression this would be called "."
- Instead of writing L(*r*) when *r* is a regular expression, we consider *r* alone to simultaneously mean both the expression *r* and the language it generates, relying on context to disambiguate

Abbreviations

- Caution: regular expressions are *strings* (programs). They are equal *only when* they contain exactly the same sequence of characters.
    - $(((1\cdot 0)\cdot(\varepsilon\cup\emptyset))^*)$ can be *abbreviated* $(10(\varepsilon\cup\emptyset))^*$
    - however $(((1\cdot 0)\cdot(\varepsilon\cup\emptyset))^*) \neq (10(\varepsilon\cup\emptyset))^*$ as strings
    - but $(((1\cdot 0)\cdot(\varepsilon\cup\emptyset))^*) = (10(\varepsilon\cup\emptyset))^*$ when they are considered to be the generated languages
        - more accurately then, $L(\ (((1\cdot 0)\cdot(\varepsilon\cup\emptyset))^*)\ ) = L(\ (10(\varepsilon\cup\emptyset))^*\ )$

          $= L(\ (10)^*\ )$

Examples

- ☐ Find a regular expression for $\{\ w \in \{0,1\}^* \mid w \neq 10\ \}$
- ☐ Find a regular expression for $\{\ x \in \{0,1\}^* \mid$ the 6th digit counting from the rightmost character of x is 1$\}$
- ☐ Find a regular expression for $L_3 = \{x \in \{0,1\}^* \mid$ the binary number x is a multiple of 3 $\}$

(*foreshadowing*: can be done by starting with DFA and then ripping states)

+ Selected examples from textbook Example 1.53 (p. 65)

Facts

- REX(Σ) is itself a language over an alphabet Γ that is

  $\Gamma = \Sigma \cup \{\ )\ ,\ (\ ,\ \cdot,\ *,\ \varepsilon\ ,\ \emptyset\}$

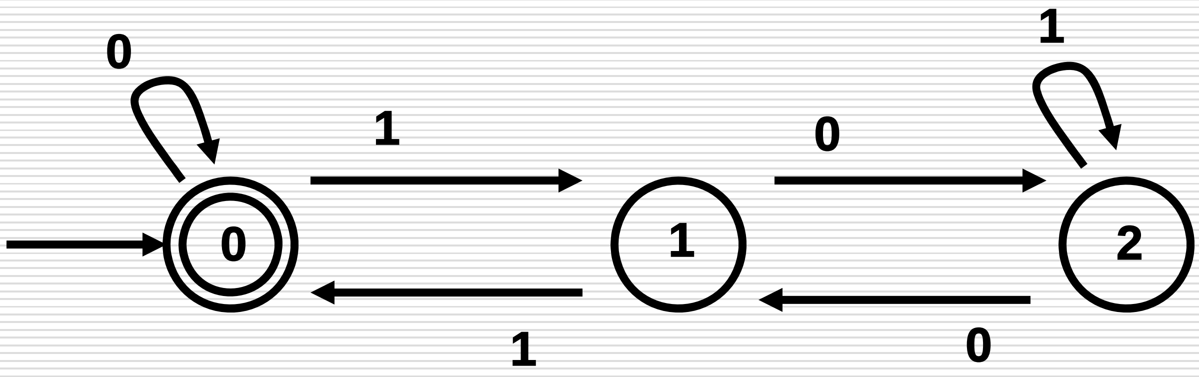
- For every Σ, |REX(Σ)| = ∞

  $\emptyset, (\emptyset^*), ((\emptyset^*)^*), \ldots$

  even without knowing Σ there are infinitely many elements in REX(Σ)

- Question: Can we find a DFA or NFA M with L(M) = REX(Σ)?

The DFA for L_3



Regular expression:
(0 ∪ 1 ____(0 1* 0)*____ 1) *

(Recall precedence of operators.)

Regular expression for L_3

- $(0 \cup 1 (0 1^* 0)^* 1)^*$
- L_3 is closed under concatenation, because of the overall form $(\ )^*$
- Now suppose $x \in L_3$. Is $x^R \in L_3$?
  - Yes: see this is by reversing the regular expression and observing that the same regular expression results
  - So L_3 is also closed under reversal

Equivalence with Finite Automata

Theorem 1.54 A language is regular if and only if some regular expression describes it.

Proof: 2 directions

Lemma 1.55: If a language is described by a regular expression, then it is regular. (Proof idea: Convert to an NFA.)

Lemma 1.60: If a language is regular, then it is described by a regular expression.

(Proof idea: Convert from DFA to GNFA to regular expression.)

Regular expressions generate regular languages

Lemma 1.55 For every regular expression r, L(r) is a regular language.

Proof by induction on regular expressions.

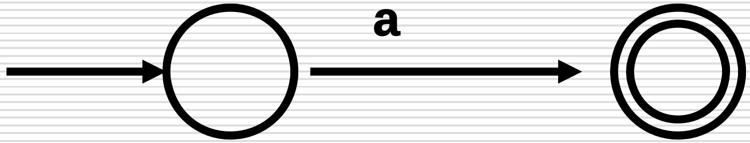
- ☐ We used induction to create all of the regular expressions and then to define their languages, so we can use induction to visit each one and prove a property about it

Recall that regular expressions were defined inductively.

L(REX) ⊆ REG

Base cases:

1. For every a∈Σ, L(a) = { a } is obviously regular:



2. L(ε) = { ε } ∈ REG also
3. L(∅) = ∅ ∈ REG

$L(REX) \subseteq REG$

Induction cases:

4. Suppose the induction hypothesis holds for r_1 and r_2. Namely, $L(r_1) \in REG$ and $L(r_2) \in REG$. We want to show that $L((r_1 \cup r_2)) \in REG$ also. But look: by definition,

   $$L((r_1 \cup r_2)) = L(r_1) \cup L(r_2)$$

   Since both of these languages are regular, we can apply Theorem 1.45 (closure of REG under ∪) to conclude that their union is regular.

$L(REX) \subseteq REG$

Induction cases:

5. Now suppose $L(r_1) \in REG$ and $L(r_2) \in REG$. By definition,
   $$L(\ (r_1 \cdot r_2)\ ) = L(r_1) \cdot L(r_2)$$
   By Theorem 1.47 (closure of REG under $\cdot$) , this concatenation is regular too.
6. Finally, suppose $L(r) \in REG$. Then by definition,
   $$L(\ \ (r^*)\ \ ) = (L(r))^*$$
   By Theorem 1.49 (closure of REG under *), this language is also regular. **QED**

On to REG $\subseteq$ L(REX)

- Now we'll show that each regular language (one accepted by an automaton) also can be described by a regular expression
  - Hence REG = L(REX)
  - In other words, regular expressions are equivalent in power to finite automata
- This equivalence is called **Kleene's Theorem** (Theorem 1.54 in book)

Converting DFAs to REX

- Lemma 1.60 in textbook
- This approach uses yet another form of finite automaton called a GNFA (generalized NFA)
- The technique is easier to understand by working an example than by studying the proof

Syntax of GNFA

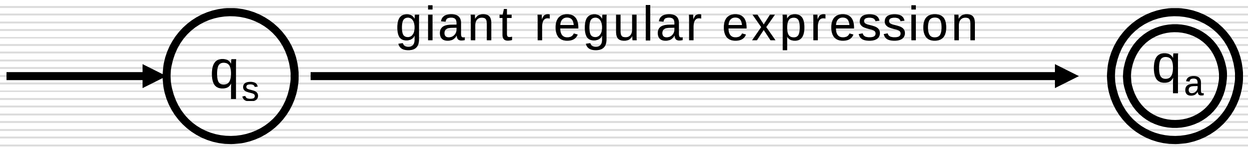
- A **generalized NFA** is a 5-tuple $(Q,\Sigma,\delta,q_s,q_a)$ such that
  1. Q is a *finite* set of states
  2. Σ is an alphabet
  3. $\delta:(Q - \{q_a\})\times(Q - \{q_s\})\rightarrow REX(\Sigma)$ is the transition function
  4. $q_s \in Q$ is the start state
  5. $q_a \in Q$ is the (one) accepting state

GNFA syntax summary

- Arcs are labeled with regular expressions
  - Meaning is that "input matching the label moves from old state to new state" -- just like NFA, but not just a single character at a time
- Start state has no incoming transitions, accept has no outgoing
- Every pair of states (except start & accept) has two arcs between them
  - Every state has a self-loop (except start & accept)

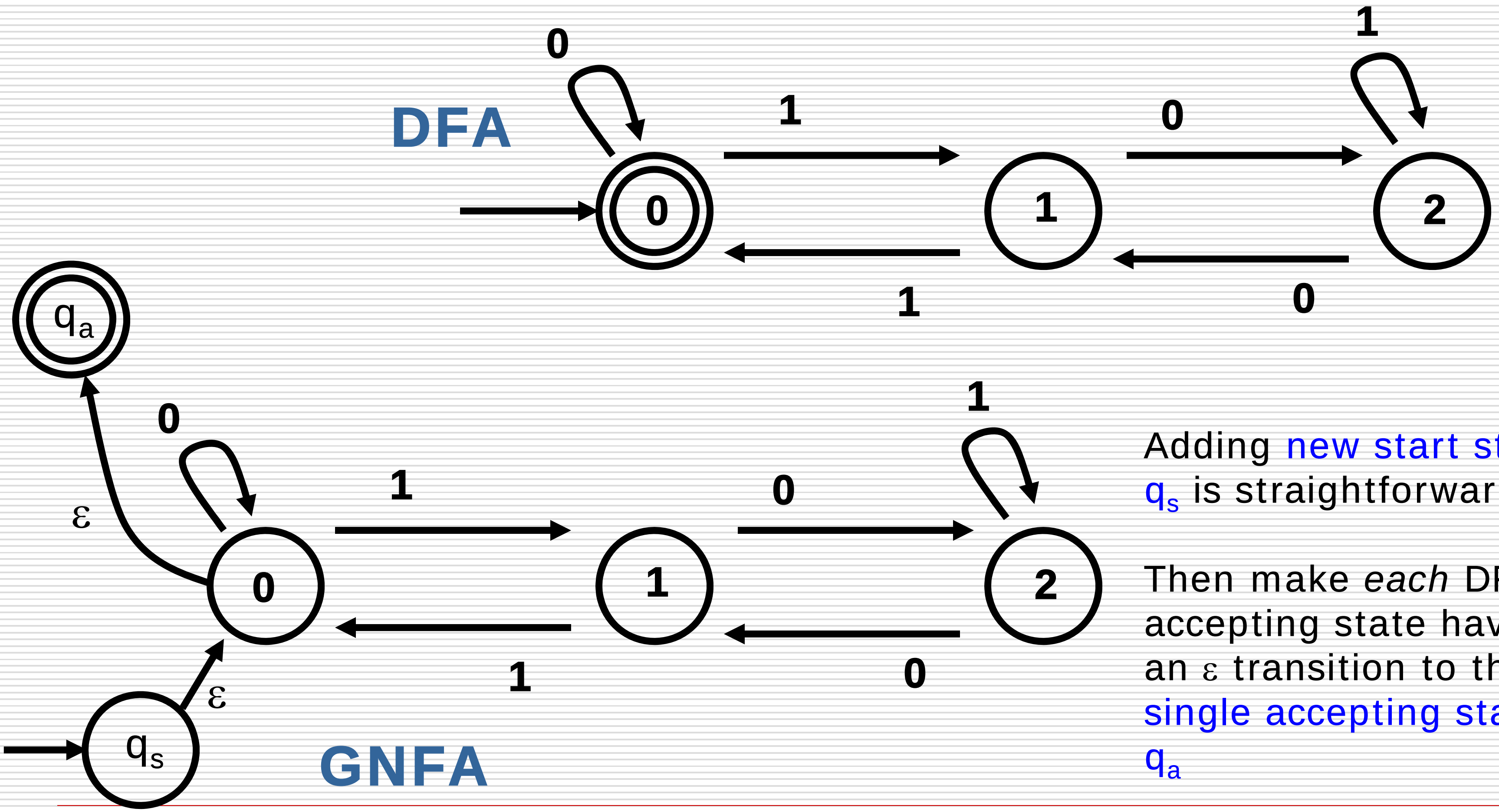
Construction strategy

- Will convert a DFA into a GNFA then iteratively shrink the GNFA until we end up with a diagram like this:



meaning that exactly that input that matches the giant regular expression is in the language

Converting DFA to GNFA



Adding new start state q_s is straightforward

Then make *each* DFA accepting state have an ε transition to the single accepting state q_a

Note: Ø transitions are not drawn here for sake of clarity, but can be important later on.

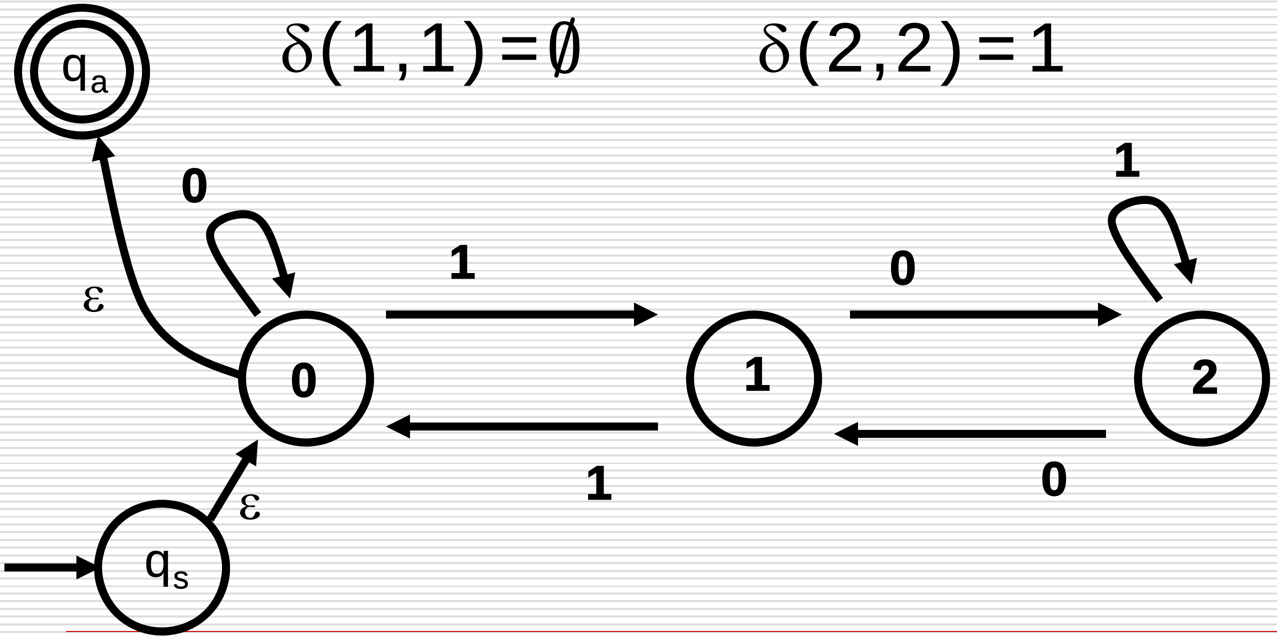
Interpreting arcs

$$\delta : (Q - \{q_a\}) \times (Q - \{q_s\}) \to \mathrm{REX}(\Sigma)$$

In this diagram, for example,

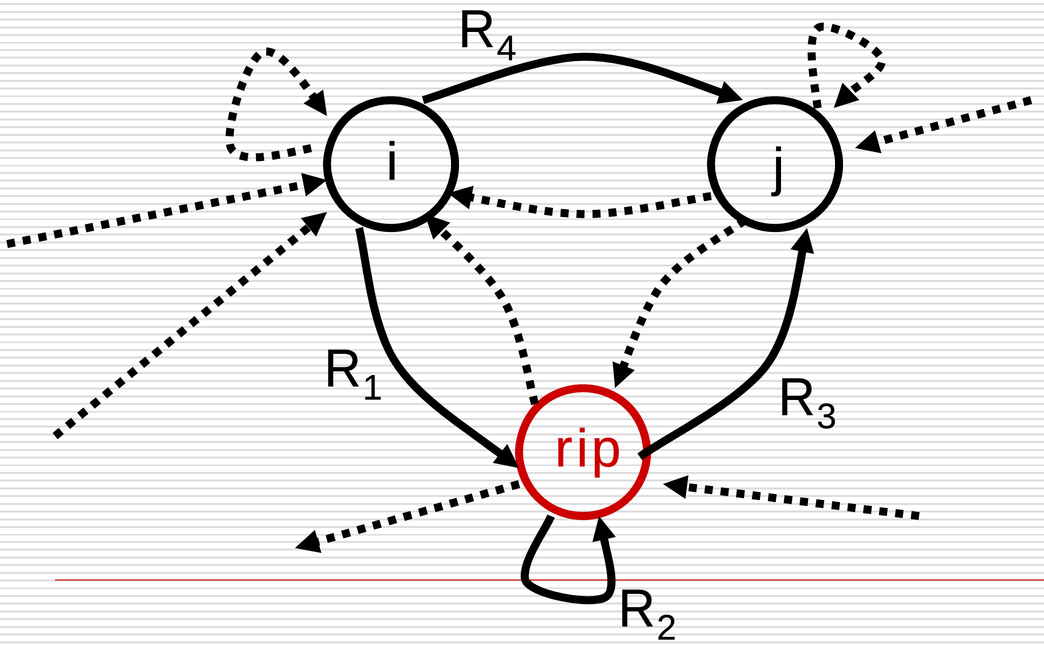
$\delta(0,1) = 1$ $\quad\delta(2,0) = \emptyset$ $\quad\delta(2,q_a) = \emptyset$

$\delta(1,1) = \emptyset$ $\quad\delta(2,2) = 1$ $\quad\delta(0,q_a) = \varepsilon$



Eliminating a GNFA state

- We arbitrarily choose an interior state (not q_s or q_a) to **rip** out of the machine

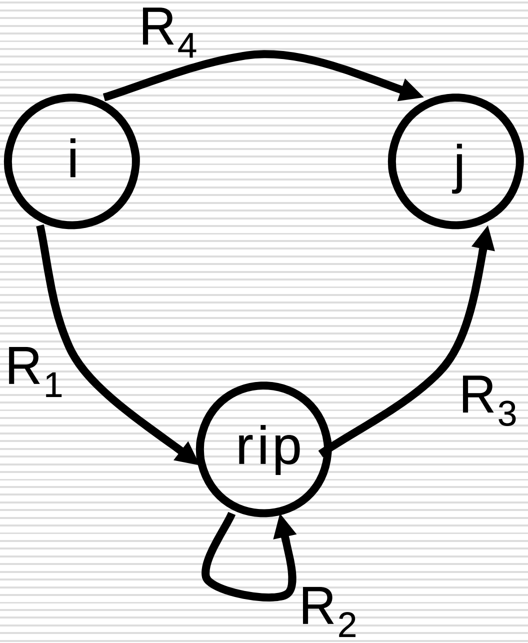


Question: how is the ability of state i to get to state j affected when we remove rip?

Only the **solid** and **labeled** states and transitions are relevant to that question

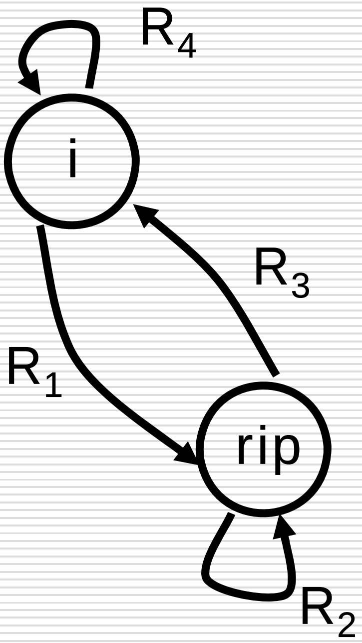
Eliminating a GNFA state

- We produce a new GNFA that omits rip
  - Its i-to-j label will compensate for the missing state
  - We will do this for **every** $(i,j) \in (Q-\{q_a\})\times(Q-\{q_s\})$
  - So we have to rewrite **every label** in order to eliminate this one state
  - New label for i-to-j is $R_4 \cup (R_1 \cdot (R_2)^* \cdot R_3)$



Don't overlook

- The case $(i,i) \in (Q-\{q_a\})\times(Q-\{q_s\})$
- New label for i-to-i is still
  $R_4 \cup (R_1 \cdot (R_2)^* \cdot R_3)$



- Example proceeds on whiteboard, but first we’ll do textbook p. 75 (Figure 1.67) for a simpler one.

g/re/p

- What does grep do?

  (int | float)_rec.*emp *becomes*
  $(\Sigma^*)(\text{int} \cup \text{float})\_\text{rec}(\Sigma^*)\text{emp}(\Sigma^*)$

- What does it mean?
- How does it work?
  - Regular expression → NFA → DFA → state reduction
  - Then run DFA against each line of input, printing out the lines that it accepts

State machines

- Very common programming technique

```
while (true) {
    switch (state) {
    case NEW_CONNECTION:
            process_login();
            state=RECEIVE_CMD;
            break;
    case RECEIVE_CMD:
            if (process_cmd() == CMD_QUIT)
                    state=SHUTDOWN;
            break;
    case SHUTDOWN:
    ...
    }
...
}
```

This chapter so far

§1.1: Introduction to languages & DFAs

§1.2: NFAs and DFAs recognize the same class of languages

§1.3: REX generates the same class of languages

- Three different programming "languages" specified in different levels of formality that solve the same types of computational problems
  - Four, if you count GNFAs

Strategies

- If you're investigating a property of regular languages, then as soon as you know L ∈ REG, you know there are DFAs, NFAs, Regexes that describe it. Use whatever representation is convenient
- But sometimes you're investigating the properties of the programs themselves: changing states, adding a * to a regex, etc. Then the knowledge that other representations exist might be relevant and might not

All finite languages are regular

Theorem (not in book) FIN $\subseteq$ REG

Proof Suppose L $\in$ FIN.

Then either $L = \emptyset$, or $L = \{ s_1, s_2, \cdots, s_n \}$ where $n \in \mathcal{N}$ and each $s_i \in \Sigma^*$.

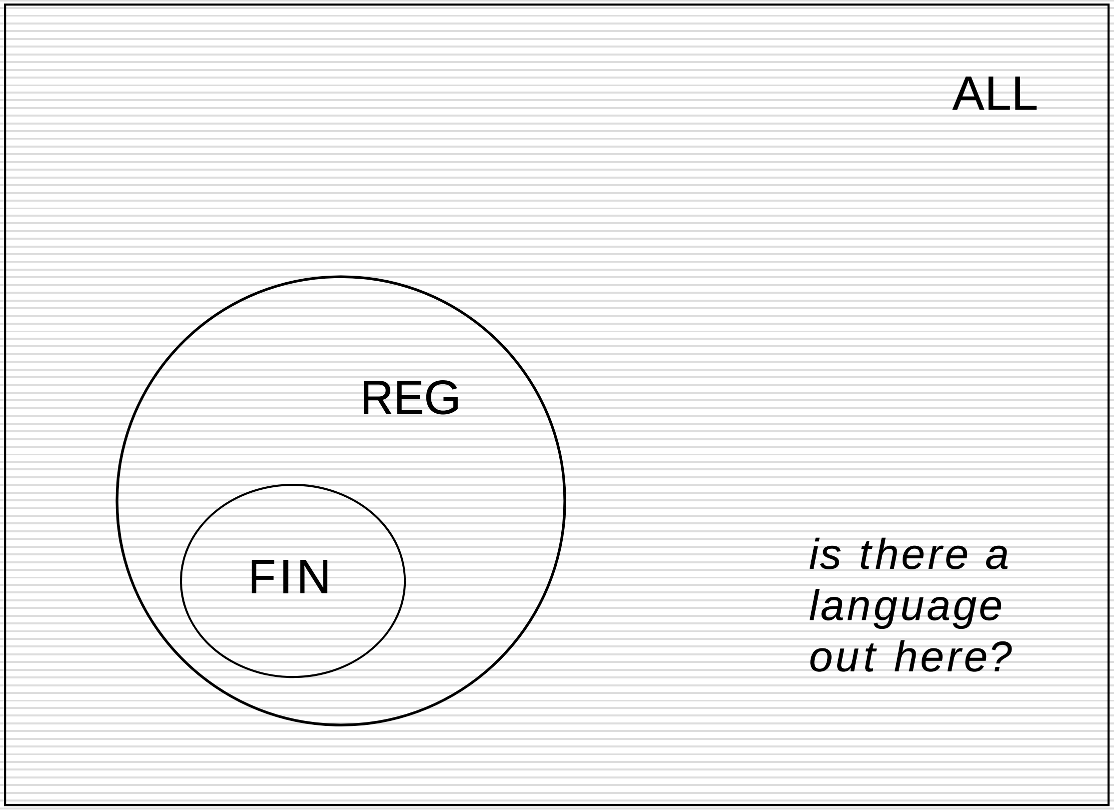
A regular expression describing L is, therefore, either $\emptyset$ or

$s_1 \cup s_2 \cup \cdots \cup s_n$ **QED**

Note that this proof does not work for $n = \infty$

Picture so far

Each point is
a language in
this Venn
diagram



ALL

REG

FIN

is there a
language
out here?

REG = L(DFA)
= L(NFA)
= L(REX)
= L(GNFA)

≠ FIN

"the class of languages generated by DFAs"