26-1/ $\overline{A_{TM}} \notin \Sigma_1$ The Halting Problem

$A_{TM} \in \Sigma_1$

$x \in A_{TM}$ iff $x = \langle M, w \rangle$
where M is a TM
$w \in L(M)$

$\overline{A_{TM}} \ni x$ iff $x \neq \langle M, w \rangle$
or $x = \langle M, w \rangle$
$w \notin L(M)$

... M does not accept w
.... M says no or loops

...
figures if M on w loops

... predicts if M "halts"

$((\Leftrightarrow))$

26-2/ $X \in \Sigma_0$ iff $X \in \Sigma_1 \wedge \overline{X} \in \Sigma_1$

$\overset{M}{X \in \Sigma_0} \Rightarrow \overset{F}{X \in \Sigma_1} \wedge \overset{G}{\overline{X} \in \Sigma_1}$:

$F = M$ $\quad G(x) = \text{not}(M(x))$

Σ_0 is closed under C(complement)

$\overset{F}{X \in \Sigma_1}$ and $\overset{G}{\overline{X} \in \Sigma_1} \Rightarrow \overset{M}{X \in \Sigma_0}$:

$M(x)$ = non-deterministically run $F(x)$ and $G(x)$

$\downarrow$ $\quad\quad$ $\downarrow$

Yes → Yes $\quad\quad$ Yes → No

26-3)

$X \in \Sigma_0 \iff X \in \Sigma_1 \land \overline{X} \in \Sigma_1$

$\neg P \iff P \to \text{False}$

$A_{TM} \notin \Sigma_0 \iff (A_{TM} \in \Sigma_0) \to \text{False}$

$\iff$ $\iff (A_{TM} \in \Sigma_1 \land \overline{A_{TM}} \in \Sigma_1) \to \text{False}$

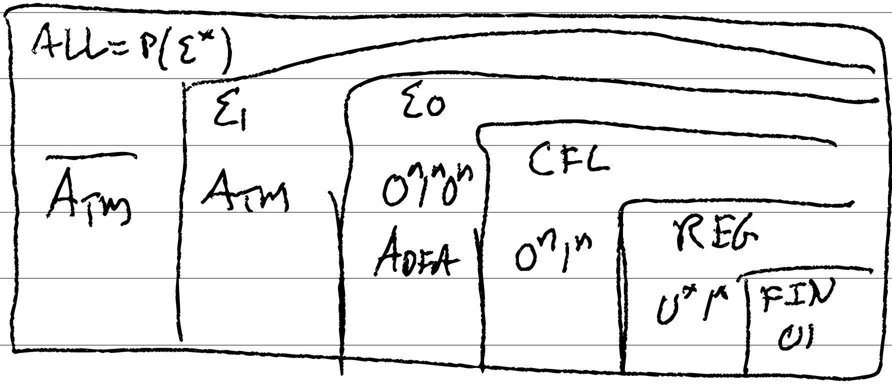
$\neg (A_{TM} \in \Sigma_1 \land \overline{A_{TM}} \in \Sigma_1)$ $\neg (P \land Q) \iff$

$\iff A_{TM} \notin \Sigma_1 \lor \overline{A_{TM}} \notin \Sigma_1$ $\neg P \lor \neg Q$

$\iff \text{False} \lor \overline{A_{TM}} \notin \Sigma_1$ $\text{False} \lor P \iff P$

$\iff \overline{A_{TM}} \notin \Sigma_1$

26-4


ALL = P(Σ*)
Σ1
Σ0
CFL
REG
$\overline{A_{TM}}$
A_{TM}
$0^n1^n0^n$
A_{DFA}
0^n1^n
0^*1^*
FIN
01

26-5/ What are the sizes of infinity?

$N = 0, 1, 2, 3, 4, \ldots$

$Z = 0, -1, 1, -2, 2, -3, 3, \ldots$

$Q = 0, 1/2, 3/4, -4/6, \ldots$

$R = 0, 1, 2, 3/4, \pi, e, \sqrt{2}, 0.\bar{3}, \ldots$

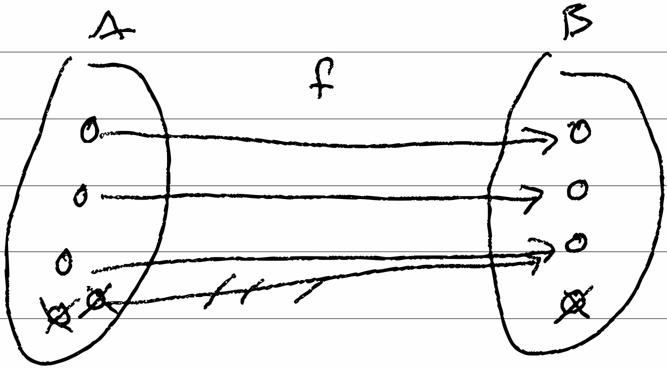
$|\{\text{kisses, hugs, puppy dogs}\}| = 3$

$|0^* 1^*|$

$|\cdot| \rightarrow$ number

set $\simeq$ set $\Rightarrow$ same size

26-6/ sets have the same size if ...



$f : A \to B$

samesize$(A, B) :=$
$\exists f : A \to B .\ o2o(f) \wedge onto(f)$

one-to-one:

$\forall x, y \in A .\ f(x) = f(y) \to x = y .$

onto:

$\forall z \in B .\ \exists x \in A .\ f(x) = z$

26-7 Natural numbers $= \{0, 1, 2, 3, 4, \ldots\}$

$\simeq$

Even numbers $= \{0, 2, 4, 6, 8, \ldots\}$

$f : \text{nat} \to \text{even}$

$f(x) = 2x$

if $A \simeq \mathbb{N}$

then A is "countable"

Georg Cantor

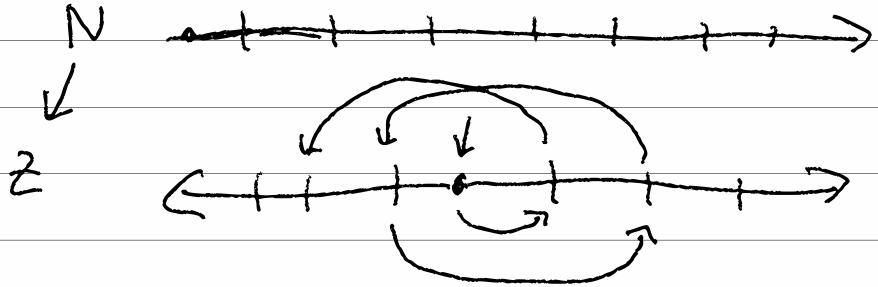
$\aleph_0$

$$2x = 2y \Rightarrow x = y$$

$\forall n \in \underbrace{\text{Evens}}. \exists x \in \text{Nat}. \; 2x = n$

$\forall y \in \text{Nat}. \exists x \in \text{Nat}. \; 2x = 2y$

26-8) $\mathbb{N} \simeq \mathbb{Z}$

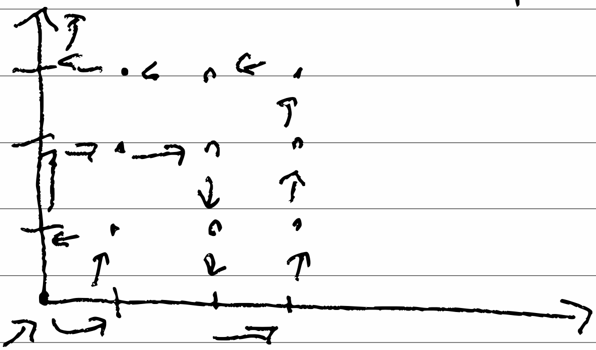


$f(0) = 0$ $\quad$ $succ(0) = 1$

$succ(+n) = -n$ $\quad$ $succ(-n) = +(n+1)$

26-9/ $\mathbb{N} \simeq \mathbb{Q} = (\mathbb{N} \times \mathbb{N}) \simeq$

$\simeq (\mathbb{Z} \times \mathbb{Z})$



Cantor pairing function $f : (\mathbb{N} \times \mathbb{N}) \to \mathbb{N}$

$f(x,y) = \frac{1}{2}(x+y)(x+y+1)+y$

2640

$$N \simeq N \times N$$

$$\simeq N \times N \times N$$

$$\begin{tikzcd}(x, y, z) \arrow[d] \\ (u, z) \arrow[d] \\ v\end{tikzcd}$$

$$N \simeq N^k \quad \forall k.$$

Fair Enumeration Combinators

Max New

26-11

$\Sigma_1 \leq N$

$\leq$

TM

$\leq$

$N \times N \times N \leq N$

to encode a TM
as a binary

$|Q| \times |\Gamma| \times$
(q_0, δ, q_a, q_r)

26-12/ Real number ... "numbers with decimals"

"weird numbers like pi"

↳ Cauchy sequences
Dedekind cuts

Numbers in binary between [0, 1)
IBS (infinite binary sequence)

$0 = .0000000 \ldots$
$1/2 = .1000000 \ldots$
$\pi/10 = \ldots$

26-13/ $IBS = \mathbb{N} \to \{0,1\}$

Is IBS countable?

$\exists f : \mathbb{N} \to IBS$. s.t.

$\text{o2o}(f) \wedge \text{onto}(f)$?

$$\neg(\exists f : \mathbb{N} \to IBS.$$
$$(\forall x, y \in \mathbb{N}.\ f(x) = f(y) \to x = y)$$
$$\wedge (\forall z \in IBS.\ \exists x \in \mathbb{N}.\ f(x) = z))$$
$$\Leftrightarrow \forall f : \mathbb{N} \to IBS.\ \neg(\text{o2o}(f) \wedge \text{onto}(f))$$
$$\Leftrightarrow \forall f : \mathbb{N} \to IBS.\ \neg\text{o2o}(f) \vee \neg\text{onto}(f)$$

26-4) $\forall f \in N \to IBS.\ \neg\, \text{onto}(f)$

$\Leftrightarrow\ \forall f \in N \to IBS,\ \neg(\forall z \in IBS.\ \exists x \in N.\ f(x) = z)$

$\Longleftrightarrow\ \forall f \in N \to IBS,$ given f.

$\exists z \in IBS = (N \to \{0,1\})$ chose z.

$\forall x \in N.$ $z(a) = \neg f(a)(a)$

$f(x) \neq z.$ given x.

must prove. $f(x) \neq z$

... $\exists b \in N.\ f(x)(b) \neq z(b)$

choose $b = x.$

$$f(x)(x) \neq z(x)$$

$$= \neg f(x)(x)$$

TRUE

26-15

i	$f(i)$
0	$0.\underline{1}101101$
1	$0.1\underline{0}1110$
2	$0.00\underline{1}1111$
3	$0.011\underline{0}110$
4	$0.1111\underline{1}11$
5	$0.110110\underline{1}10$

$z =$

$0.010101\ldots$

Cantor's Diagonalization Proof

26-16

$$\Sigma_1 < N < IBS \simeq ALL$$

$$\Sigma_1 < ALL$$

$ALL = P(\Sigma^*)$

$= P(\{\overset{\nearrow 0}{\varepsilon}, \overset{\nearrow 1}{0}, \overset{\nearrow 2}{1}, \overset{\nearrow 3}{00}, \overset{\nearrow 4}{01}, \overset{\nearrow 5}{10}, \overset{\nearrow 6}{11}, \overset{\nearrow 7}{000}, \ldots\})$

$= \{ \emptyset, \{\varepsilon\}, \{0\}, \{000\},$

$\{\varepsilon, 0\}, \{\varepsilon, 0, 00\}, \ldots$

$\ldots \}$

$ALL = IBS$

elements of ALL are subsets of Σ^*

$\emptyset = 0$ to everything

$f(x) = 0$

$f : ALL \rightarrow IBS$

$f(A) = \lambda i.\ lexi(i) \in A.$