?5-1

$BT = \text{Leaf } () \mid \text{Branch } (\text{Tree}, \text{Num}, \text{Tree})$

$BT = \mu B.\ 1 + (B \times (\text{Num} \times B))$

```
mt : BT = Leaf ()
insert (b:BT) (n:num) : BT :=
  case b with
     Leaf () => Branch (b, n, b)
     Branch (l,x,r) => if n ≤ x then
                          Br (insert l n), x, r
                          o.w. Br l, x, (insert r n)
mem (b:BT) (n:num) : bool :=
  case b with   Leaf () => false
             Br (l,x,r) => if x == n then true
                           if n < x then mem l n
                           o.w.          mem r n


inv : BT := Br (Br (Leaf (), 7, Leaf ()), 3, Leaf ())
   mem inv 7 = false
```

$\forall b \in BT, n \in Num.\ \text{mem (insert } b\ n)\ n = \text{true}$

$BST_{l\,r} := \mu B.\ 1 + (\Pi(n:\text{num} \mid l \le n \le r) \times (BST\ l\ n) \times (BST\ n\ r))$, with $l:\text{num}$, $r:\text{num}$

F-bounded polymorphism   dependent product

```
client := ΛB. λ(i : (B × (B × Num → B) × (B × num → bool)).
             let mt = fst i ;
                 ins = fst snd i ;          (client [BT]
                 mem = snd snd i ;             (pair mt (pair insert
             in                                             mem)))
              ...
```

25-2/ server := $\Lambda C.\ \lambda(cl : \forall B.\ (B \times (B \times n \to B) \times (B \times n \to bool)) \to C)$

$cl[BT]\ (\ldots\ldots)$

$T ::= \ldots\ldots \mid \exists A.\ T$ BST : $\exists B.\ (B \times (B \times N \to B) \times (B \times N \to B$

$M ::= \ldots\ldots \mid \text{pack}\ [A = T]\ M\ \text{as}\ T'$

$\mid \text{unpack}\ [A]\ X\ \text{from}\ M\ \text{in}\ M'$

$V ::= \ldots\ldots \mid \text{pack}\ [A = T]\ V\ \text{as}\ T'$

$E ::= \ldots\ldots \mid \text{pack}\ [A = T]\ E\ \text{as}\ T'$

$\mid \text{unpack}\ [A]\ X\ \text{from}\ E\ \text{in}\ M$

$E[\ \text{unpack}[A]\ X\ \text{from}\ (\text{pack}\ [A' = T]\ V\ \text{as}\ T)\ \text{in}\ M\ ]$

$\Rightarrow E[\ M[X \leftarrow V]\ [A \leftarrow T]\ ]$

$$\frac{\Gamma \vdash M : T[A \leftarrow T']}{\Gamma \vdash \text{pack}\ [A = T']\ M\ \text{as}\ T : \exists A.T}$$

$\left(\begin{array}{l}\text{pack}\ [B = \mu B.1 + (B \times N \times B)] \\ \quad \text{insert}\ \text{as}\ (B \times N \to B)\end{array}\right) : \exists B.\ B \times N \to B$

$$\frac{\Gamma \vdash M_1 : \exists A'.\ T' \qquad \Gamma, X \to T'[A' \leftarrow A] \vdash M_2 : T}{\Gamma \vdash \text{unpack}\ [A]\ X\ \text{from}\ M_1\ \text{in}\ M_2 : T}$$

unpack $[A]$ ins from in

42 : num

pack $[HR = \mu B.\ 1 + (B \times N \times B)]$

let prive ...... in

(mt, insert, mem) as $(HR \times (HR \times N \to HR) \times (HR \times N \to Bool))$

$: \exists R.\ (R \times (R \times N \to R) \times (R \times N \to Bool))$

5-3

$f : D \rightarrow R$

$f \quad c$ where c isn't exactly D, but related

$D = \text{Animal} \qquad c = \text{Cat} \qquad \text{Cat} \subseteq \text{Animal}$

$\forall T_2.\ T_1 \times T_2 \supseteq \forall T_2, T_3.\ T_1 \times (T_2 \times T_3)$

struct posn { int x; int y; }
struct trio { $\underbrace{\text{struct posn p;}}_{\text{int x; int y;}}$ int z; }

Records

$M ::= \ldots \mid \langle L_1 = M_1, \ldots, L_n = M_n \rangle \mid M.L$

$L ::=$ some set

$V ::= \ldots \mid \langle L_1 = V_1, \ldots, L_n = V_n \rangle$

$E ::= \ldots \mid \langle L_1 = V_1, \ldots, L_i = E, L_{i+1} = M_{i+1}, \ldots L_n = M_n \rangle$
$\mid E.L$

$E[\langle L_1 = V_1, \ldots, L_n = V_n \rangle . L_i] \Rightarrow E[V_i]$

$T ::= \ldots \mid \langle L_1 = T_1, \ldots, L_n = T_n \rangle$

$$\frac{\Gamma \vdash M : \langle L_1 = T_1, \ldots, L_n = T_n \rangle}{\Gamma \vdash M.L_i : T_i}$$

$$\frac{\Gamma \vdash M_1 : T_1 \quad \vdots \quad \Gamma \vdash M_n : T_n}{\Gamma \vdash \langle L_1 = M_1, \ldots, L_n = M_n \rangle : \langle L_1 = T_1, \ldots, L_n = T_n \rangle}$$

25-4

$$\left(\lambda(p : \langle x : num, y : num \rangle).\ \sqrt{p.x^2 + p.y^2}\right)\left(\langle x : 5, y : 4, z : 3 \rangle\right)$$

$$\frac{\Gamma \vdash M : D \to R \quad \Gamma \vdash N : D}{\Gamma \vdash M\ N : R}$$ ← rejects, non-stuck prog

new relation

$$\frac{\Gamma \vdash M : D_1 \to R \quad \Gamma \vdash N : D_2 \quad D_2 \le D_1}{\Gamma \vdash M\ N : R}$$ $\le$ = subtype

$\underbrace{\langle x, y, z \rangle}_{\text{smaller}} \le \underbrace{\langle x, y \rangle}_{\text{big}}$

$\langle L_1 = T_1, \ldots, L_n = T_n \rangle$ = Left

$\le$

$\langle L'_1 = T'_1, \ldots, L'_n = T'_n \rangle$ = Right

iff Right $\subseteq$ Left

$T \le T$

$(f\ (g : D_y \to R_y)$

$\ldots \to D_y$ $\quad b : R_y \to \ldots$

$b\ (g\ a)\ )$

$(h : D_x \to R_x)\ )$

$D_y \le D_x$ [argu more specific]

$R_x \le R_y$

$(x : D_x \to R_x) \le (y : D_y \to R_y)$

give!? Animal → Bool ok

want: Cat → Bool

got : Ani → Ani

want : Cat → Cat

Liskov Substitution Principle: $x : X$ and $y : Y$, $X \le Y$

iff $\forall C.\ (\exists t.\ C[y] : T) \Rightarrow (\exists t.\ C[x] : T)$

Structural sub-typing

nominal sub-typing (by name) — Java

F-bounded Polymorphism

old : $\forall A . T$ $\quad$ new: $\forall A \le T.\ T'$

$$\frac{\Gamma \vdash M : \forall A \le T'.\ T'' \quad T \le T'}{\Gamma \vdash M[T] : T''[A \leftarrow T]}$$