23-1/

$A_{TM} = \{ \langle M, w \rangle \mid M \in TM \text{ and } M \text{ accepts } w \}$

$\langle$ binary add , $0110 + 1000 = 1110 \rangle \in A_{TM}$

$\langle$ palindrome , $0110 \rangle \in A_{TM}$

$\langle$ palindrome , $1000 \rangle \notin A_{TM}$

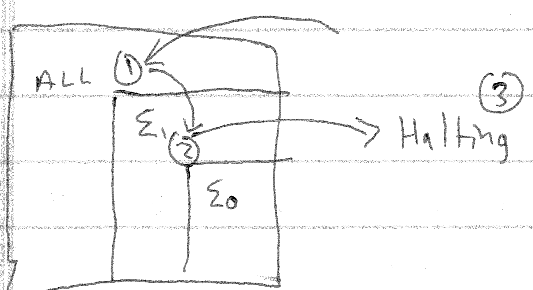
Σ_1 : On input $\langle M, w \rangle$,
- use tape2 as w
- use tape3 for q (initialized q_0)
- use tape4 for δ
- simulate the machine

Σ_0 ++++ $\underbrace{\text{if } M \text{ diverges on } w}_{\text{The Halting Problem}}$, then reject

$X \notin \Sigma_0$ where X is the Halting Problem

"Turing Pumping Property" $X \notin REG \Leftarrow \neg RPP(X)$

$+ \ \forall x \in REG.\ RPP(x)$



$\exists X.\ X \in ALL \wedge X \notin \Sigma_1$

$\Sigma_1 \subseteq ALL$

Prove that ALL is bigger

$|\{x, y, z\}| = 3$ $\quad |\bullet|$: set $\rightarrow$ natural

$|\{x, y\}| = 2$ $\quad$ cardinality function

matching elements $\Rightarrow$ same size

non-matching elements $\Rightarrow$ diff (ie one is bigger)

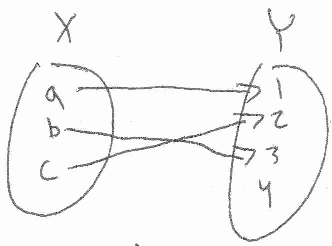
$(X \leq Y) \wedge (X > Y \vee Y > X)$ $\quad X = \Sigma_1$

$\Rightarrow Y > X$ $\quad Y = ALL$

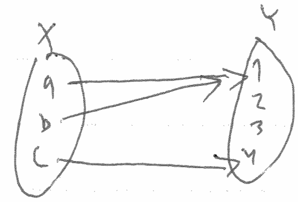
23-2/ Set X and Set Y have the same size if there exists a one-to-one and onto function, m, from X to Y.

> one-to-one: $\forall a, b \in X.\ m(a) \neq m(b)$ if $a \neq b$

$m(a) = m(b) \Rightarrow a = b$

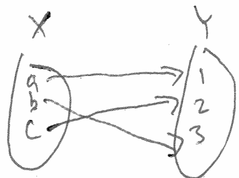


Ok

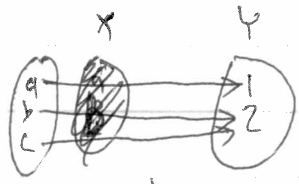


BAD

> onto: $\forall b \in Y.\ \exists a \in X.\ m(a) = b$



Ok



ok

The set of naturals N is the same size as the set of Even numbers, E

$N = 0, 1, 2, 3, 4, 5, 6, 7, \ldots$

$E = 0, 2, 4, 6, 8, 10, 12, 14, \ldots$

$m(n) = 2 \times n$

① $m(a) = m(b) \Rightarrow a = b$

$2a = 2b \Rightarrow a = b$

$a = b \Rightarrow a = b$

② $\forall b \in E.\ \exists a \in N.\ m(a) =$

$\exists a \in N.\ m(a) = 2 \times n$

$\exists a.\ 2 \times a = 2 \times n$

$a = n$

$\aleph_0$ or "countable"

or "same size as naturals"

$N \times \{a, b\} = \{ (0,a)\ (0,b)\ (1,a)\ (1,b) \ldots \}$

$m(n, a \vee b) = 2n$ if a

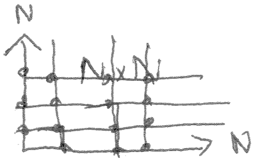
$2n + 1$ if b

Z are countable

$N \times F$ where F is finite, then it is countable

23-3/

$N \times N = \{(0,0)\ (1,1)\ (0,1),\ (2,3)\ \dots\}$



$N \times N$ is the same size as N

plane line

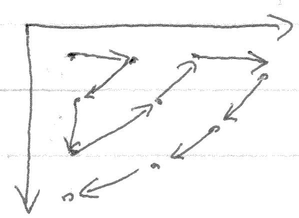
$m((a,b)) = m((x,y))$ = one number

$m(x,0) = x$ BAD not 1-1

$m(0,y) = y$ BAD

$m(x,y) = x+y$ BAD

$m(x,y) = 2^x \cdot 3^y$ BAD not onto



$m(x,y) = \frac{1}{2}(x+y)(x+y+1) + y$

Q (~~integers~~ rationals) are countable

$N \times N \simeq N$

$N \times (N \times N) \simeq N \times N \simeq N$

$N^k \simeq N$

$A \simeq N \wedge B \simeq N \Rightarrow A \cup B \simeq N$

The set of Turing machines over $\Sigma = \{0,1\}$, $\Gamma = \Sigma \cup \{\sqcup\}$

$\langle Q, \Sigma, \Gamma, q_0, \delta : Q \times \Gamma \to Q \times \Gamma \times \{L,R\}, q_a, q_r \rangle$

$Q \quad |Q| \quad (Q \times \Gamma) \times (Q \times \Gamma \times \{L,R\}) \quad |Q| \quad |Q|$

$n = |Q| \quad n \times (n \times 3) \times (n \times 3 \times 2) \times n \times n$

$= 18 \times n^5$

($N \to$ Finite set)

$0 \to 0$

$1 \to 18$

TMs $\simeq N$

$\Sigma_0 \simeq N$

$\Sigma_1 \simeq N$

23-4/ An infinite binary sequence, B, is a function from
N to 0 or 1.

$B_0(n) = 0$ $\qquad$ $B_1(n) = 1$ $\qquad$ $B_{10}(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

000000... $\qquad$ 1111... $\qquad$ 0101010101...

$(X \simeq Y$ then $Y \simeq X)$

Let's assume that $B \simeq N$.

$\exists m : B \to N$ where ~~B~~ is 1-1 and onto

① $\forall a, b \in B.\ m(a) = m(b) \Rightarrow a = b$

② $\forall n \in N.\ \exists a.\ m(a) = n$

$\neg$② $= \exists n \in N.\ \forall a.\ m(a) \neq n$

$\neg$① $= \exists a, b \in B.\ m(a) = m(b) \Rightarrow a \neq b$

$m(B_0) = 12$ $\qquad$ $m(B_1) = 13$

$m(B?) = 0$ $\qquad$ $m(B??) = 1$

N	B
0	0111000111 0..
1	11 0011000000..
2	00 0 1 1 1 1 1
3	0011
4	
5	

Assume $N \simeq B$ $\qquad$ $m(12) = B_0$

$\exists m : N \to B$ $\qquad$ $m(13) = B_1$

① $\forall a, b \in N.\ m(a) = m(b) \Rightarrow a = b$

② $\forall n \in B.\ \exists a.\ m(a) = n$

$\neg$② $\exists n \in B.\ \forall a.\ m(a) \neq b$

$n =$ the missing binary string $= (\text{fun } i \Rightarrow \neg\, m(i)(i))$

$m(i) =$ the ith binary string

$m(i)(i) =$ the ith digit of the ith binary string

$= 0101\ldots$

$\neg(m(i)(i)) = 1010\ldots$

$\forall a \in N.\ m(a) \neq (\text{fun } i \Rightarrow \neg\, m(i)(i))$

take any a ⊢ $m(a) \neq (\text{fun } i \Rightarrow \neg\, m(i)(i))$

~~m(a)(a) =~~

$m(a)(a) \neq (\text{fun } i \Rightarrow \neg\, m(i)(i))(a)$

$m(a)(a) \neq \neg\, m(a)(a)$

$0 \neq \neg 0$

$0 \neq 1$