26-1

Countable $\neq$

$N = \Sigma^*$ Reals

$\shortparallel$ $\shortparallel$

$N \times N$ (Q) R_{01} $[0,1)$

$\shortparallel$ $\shortparallel$

N^k set of Infinite binary sequence

$\shortparallel$ IBS $= N \to \{0,1\}$

TM (11) — TODAY

$\shortparallel$ ALL $= X_1$

Σ_1

$\forall \Sigma$. (Assume $\Sigma = \{0,1\}$)

$x \in$ ALL ALL $= P(\Sigma^*)$

$x = \{01, 111, 000\} \in$ ALL

$\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11,$
$000, 001, 010, 011,$
$100, 101, 110, 111, \dots\}$

I wrote in "lexicographic ordering"
= strings of length i appear before len j whenever $i < j$
within length i, strings with lower binary values appear first

$\Sigma^* \approx N$ (is countable) (lexico-order is bijection)

$bi(0) = \varepsilon$ $bi(1) = 0$ $bi(4) = 01$

$bi^{-1}(111) = 14$

$P(\{a,b,c\}) = \{\{\}$ $\{a\}$ $\{b\}$ $\{c\}$ $\{a,b\}$ $\{a,c\}$
$\{b,c\}$ $\{a,b,c\}\}$

$= \{000$ 100 010 001 110 101
011 $111\}$

$P(X)$ = binary strings of length $|X|$

26-2)

$ALL := N \to \{0, 1\}$

where $x \in ALL$

$w \in x$ iff $x(\text{lexico-ordering}^{-1}(w)) = 1$

$w \notin x$ iff $x(\text{lexic-ordering}^{-1}(w)) = 0$

$\{\varepsilon, 0, 11, 010\} = x \in ALL$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$0 \quad 1 \quad 6 \quad 9$

$x(pos) = 1$ if $pos \in \{0, 1, 6, 9\}$

0 o.w

ALL is a set of languages

$x \in ALL$ is a language (ie a set of strings)

language := function that says yes on position

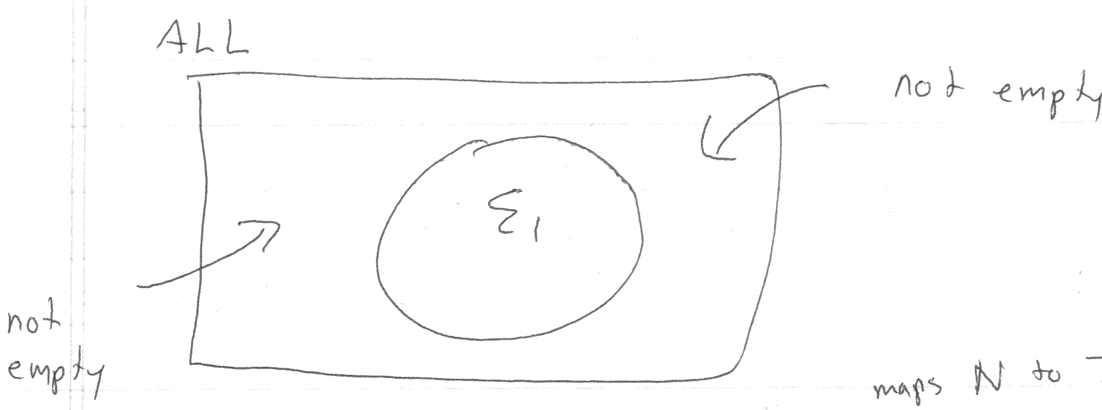
setoflanguage := set of functions

$x(pos) = 1$ if pos is even

0 ow.

$x = \{\varepsilon, 1, 01, 11, 001, 011, \ldots\}$

$= \varepsilon \cup \{w \mid w$ is an odd binary number$\}$



maps N to TMs

language $W = \lambda(pos) . \neg(\Sigma_1(pos))(pos)$

W says whatever TM T does when given T, do the opposite

5-3

$$H(\langle M, w\rangle) = \begin{cases} \text{acc} & \text{if } M \text{ accepts } w \\ \text{rej} & \text{if } M \text{ does not accept } w \end{cases}$$

D = " On input $\langle M\rangle$, where M is a TM:

1. Run H on $\langle M, \langle M\rangle\rangle$ $= W$
2. Output the opposite of H "

$$D(\langle M\rangle) = \begin{cases} \text{acc} & \text{if } M \text{ does not accept } \langle M\rangle \\ \text{rej} & \text{if } M \text{ \underline{does} accept } \langle M\rangle \end{cases}$$

$$D(\langle D\rangle) = \begin{cases} \text{acc} & \text{if } D \text{ does not accept } \langle D\rangle \\ \text{rej} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

if $D(\langle D\rangle) = \text{acc}$, then $D(\langle D\rangle)$ must $\neq$ acc

$D(\langle D\rangle) \neq \text{acc}$, then $D(\langle D\rangle)$ must $=$ acc

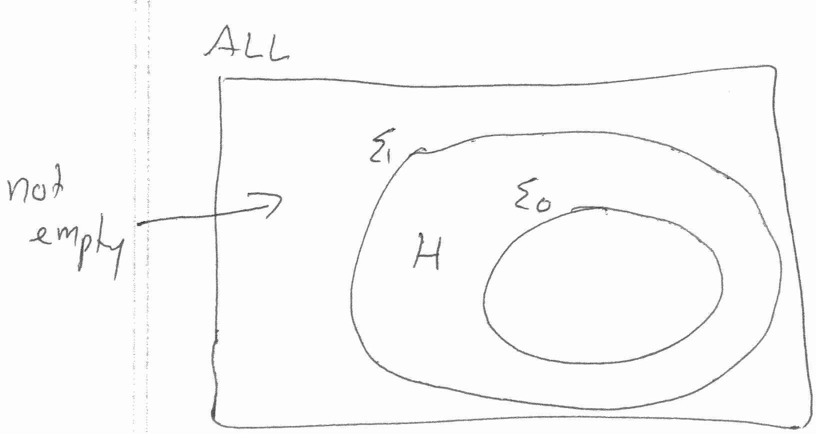
D is a Quine Liar's Paradox :

"This statement is false."

The only thing D can do is <u>diverge</u>

therefore H must diverge sometimes

therefore $H \in \Sigma_1$ but not $\in \Sigma_0$



26-4

Lemma: $(A \text{ iff } B) = (A \Rightarrow B) \wedge (B \Rightarrow A)$

$A \in \Sigma_0$ iff $A \in \Sigma_1$ and $\bar{A} \in \Sigma_1$

Foward: Assume $A \in \Sigma_0$, prove others

$A \in \Sigma_1$, trivial

$\bar{A} \in \Sigma_1$, easy = "run $A(w)$, negate answer"

Backwards: Assume $A \in \Sigma_1$ and $\bar{A} \in \Sigma_1$

Prove $A \in \Sigma_0$

Given machine YESA (can diverge when $w \notin A$) NOA (can diverge when $w \in A$)

make ALWAYSA (can never diverge)

ALWA (w) = interleave execution of YESA(w) and NOA(w)

if Y says ACC, we ACC

if N says ACC, we REJ

$(A \text{ iff } B) \Rightarrow (\neg A \text{ iff } \neg B)$

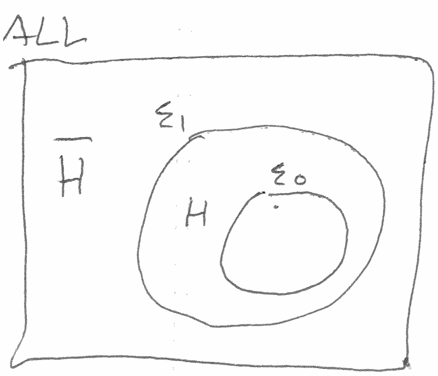
$H \notin \Sigma_0$ iff $\neg(H \in \Sigma_1$ and $\bar{H} \in \Sigma_1)$

$\neg(A \wedge B) \Rightarrow \neg A \vee \neg B$

$\Downarrow$

$H \notin \Sigma_1$ OR $\underbrace{\bar{H} \notin \Sigma_1}_{\text{must be true}}$

we know $H \in \Sigma_1$



H — Universal Turing Machine

H is "undecidable"

$\bar{H}$ is "unrecognizable" or "uncomputable"

$\bar{H}$ — Halting Problem

6-5

Mapping Reducible ($A \leq_m B$)

$\exists f.\ \forall w \in \Sigma^*.\ w \in A$ iff $f(w) \in B$

f translates an A problem into a B problem

If $A \leq_m B$ and $B \in \Sigma_0$, then $A \in \Sigma_0$

If $A \leq_m B$ and $A \in \Sigma_1$, then $B \in \Sigma_1$

A is "big" B is "small"

A can't work so B can't work

$H \leq_m E_{TM} = \{ \langle M \rangle \mid M \in TM \text{ and } L(M) = \emptyset \}$

$f(\langle M, w \rangle)$ = "On input x,
if $x = w$, then simulate M on w
o.w. reject"

$H \leq_m REG_{TM} = \{ \langle M \rangle \mid M \in TM \text{ and } L(M) \in REG \}$

= "On input x,
accept if $x \in 0^n 1^n$
OR if M accepts w"

TM variant called LBA

linear-bounded automata

"A TM with a finite tape"

they can only LOOP not DIVERGE

↓ return to same states

↓ always visit new states

H_{LBA} is decidable by an LBA

$|\Gamma|^{|w|} \times |Q| \times |w|$

tapes states head-pos

$E_{LBA} \notin LBA$

ALL_{CFG}