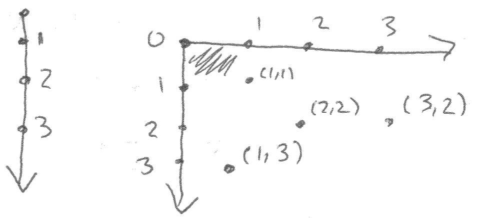
25-1

N and E are the same size
naturals evens

N and Q (rationals) $\longrightarrow$ 1/4, 4/8, 8/12, 1/3, etc
$\hookrightarrow$ 0, 1, 2, 3, 4, etc

$Q = N \times N$

N Q $\frac{a}{b} \in Q$ iff $(a, b) \in N \times N$



N ss Q?
$\exists f$ where f is a bijection from N to Q
where f^{-1} is a bijection from Q to N

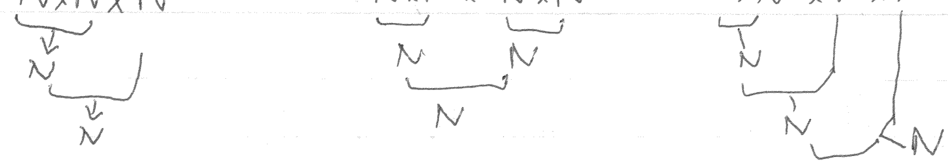
X $f(n) = (0, n) : N \to Q$

X $f^{-1}(x, y) = x + y : Q \to N$

✓ $f^{-1}(x, y)$ = assume $X = x_0 x_1 \dots x_n$ where $x_i \in \{0, 1\}$
$y = y_0 y_1 \dots y_m$
$x_i = 0$ if $i > n$
$y_i = 0$ if $i > m$
$l = \max(n, m) \times 2$
return $z = z_0 \dots z_l$ where $z_{2n} = x_n$, $z_{2n+1} = y_n$

✓ $f^{-1}(x, y)$ = cut ~~diag~~ diagonal strips
$\frac{1}{2}(x+y)(x+y+1) + y$ $\leftarrow$ Cantor's Function

N ss $N \times N \times N$ $N \times N \times N \times N$ $N \times N \times N \times N$



$N \times N \times N \to N$

N ss N^k, $\forall k$

$Z = \{+, -\} \times N$

if X is countable or finite
and Y is countable
then $X \times Y$ is countable

Turing Machines are COUNTABLE. $TM = (Q, \Sigma, \Gamma, q_0, \delta, q_a, q_r)$
$P(F) \times F \times F \times Q \times F \times Q \times Q$

25-2/ Is there a set that's infinite and NOT the same size as $\mathbb{N}$?

$\mathbb{R}$ = real numbers

$0 \in \mathbb{R}$ $\quad 42 \in \mathbb{R}$ $\quad \pi \in \mathbb{R}$

$e \in \mathbb{R}$ $\quad \sqrt{2} \in \mathbb{R}$

↳ irrational (ie $\lnot \exists x, y.\ \pi = x/y$)

↳ infinite digits

$\mathbb{R} \neq \mathbb{N} \times \mathbb{N}$ (not rational!)

$\mathbb{R} \neq \mathbb{N}^*$ (not finite)

$\mathbb{R} = P(\mathbb{Q}) \times P(\mathbb{Q})$ — if $(S, G) \in \mathbb{R}$, then $\exists$ real r where $\forall s \in S.\ s < r$, $\forall g \in G.\ r < g$

Dedekind Cut

"Those $\delta - \varepsilon$ proofs, huh?"

$\mathbb{R}$ = Cauchy sequence (infinite series of $\mathbb{Q}$, converging to the r)

C.S-y way $\mathbb{R}$s between $[0, 1) = \mathbb{R}_{01}$ (write number in binary)

$\mathbb{R}_{01} = \mathbb{N} \to \{0, 1\}$

$.5 \simeq .10\ldots = \lambda pos.$ if $pos = 0$, then 1
o.w., then 0

$.75 \simeq .110\ldots = \lambda pos.$ if $pos = 0$, then 1
$pos = 1$, then 1
o.w, 0

$.\overline{10}$
$= \lambda pos.$ if pos is even, 1
o.w., 0

$\pi/4 = \lambda pos. \ \ldots ?$

$\exists f \in$ bijection from $\mathbb{N}$ to $\mathbb{R}_{01}$?

$f(n) = r_n$ $\quad r_n = \lambda pos. \ldots ?$

$f^{-1}(r_n) = n$

onto: $\forall r \in \mathbb{R}_{01}.\ \exists n \in \mathbb{N}.\ f(n) = r$

$f(22) = .\overline{10}$

!onto: $\exists r \in \mathbb{R}_{01}.\ \forall n \in \mathbb{N}.\ f(n) \neq r$

$f(2048) = .101101$

$\boxed{\exists r \in \mathbb{R}_{01}.\ \forall n \in \mathbb{N}.\ \cancel{\forall pos.}\ \exists pos.\ f(n)(pos) \neq r(pos)}$

$f(99) = \pi/4$

$r = \lambda pos.\ \neg f(pos)(pos)$

$f(pos) =$ some r (r_J)

$r_J(pos)$

5-3

$\mathbb{N}$ is $\aleph_0$ (aleph 0)

$\mathbb{R}$ is $\aleph_1$

$\mathbb{R}$ is bigger than $\mathbb{N}$

$\downarrow$ $\qquad$ $\downarrow$

$\mathbb{R}_{01}$ $\qquad$ $\mathbb{N} \times \mathbb{N}$

$\downarrow$ $\qquad$ $\downarrow$

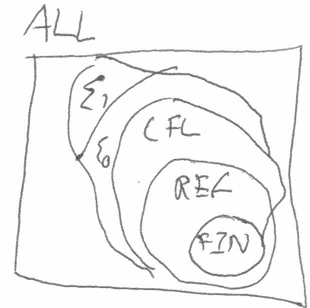
Infinite binary sequence $\qquad$ $F \times \mathbb{N}$

$\mathbb{N} \Rightarrow \{0,1\}$ $\qquad$ $\downarrow$

$\qquad$ $F^k \times \mathbb{N}^j$

$\downarrow$ $\qquad$ $\downarrow$

ALL $= \boxed{?}$ is bigger than TM



ALL $= \Sigma_1$

$\Rightarrow$ all problems are solvable by TMs

Next time!